LINEAR FILTERING FOR VASICEK TERM STRUCTURE MODEL WITH SEQUENTIALLY CORRELATED NOISE

Wu Shu, Liu Sifeng

(College of Economics and Management, NUAA, 29 Yudao Street, Nanjing, 210016, P.R. China)

Abstract: When Kalman filter is used in the estimation of Vasicek term structure of interest rates, it is usual to assume that the measurement noise is uncorrelated. Study results are more favorable to the assumption of correlated measurement noise. An augmented state Kalman filter form for Vasicek model is proposed to optimally estimate the unobservable state variable with the assumption of correlated measurement noise. Empirical results indicate that the model with sequentially correlated measurement noise can more accurately describe the dynamics of the term structure of interest rates.

Key words: Vasicek term structure model; augmented Kalman filter; sequentially correlated noise; state estimation

CLC number: F830.9 **Document code:** A **Article ID:** 1005-1120(2011)03-0309-06

INTRODUCTION

Term structure models are one of the key metrics in financial analysis. The studies on these models are the key to understand monetary policy effects and its transmission mechanism[1]. Dynamics modeling of the term structure of interest rates refers to two problems: One is, in terms of cross-section dimension, to depict the relationship between zero-coupon bond yields and the time to maturities; The other is, in terms of time series dimension, to specify the dynamics of the term structure over time, which are mainly the unobservable state variables. The two dimension data constitute panel data.

Some references have focused on the affine class of term structure models because of their analytic tractability[2-3]. In the context of affine models, the yields to maturity are affine functions of state variables. An effective and extensively used methodology to optimally estimate unobservable state variables from noisy panel data is Kalman filter[4]. Examples especially in the affine area are Refs. [5-6], etc. In most references elaborating on term structure models using estimation technique of Kalman filter, measurement noise is assumed to be independent and identically distributed (i. i. d) for computational convenience. However, when allowing for a general measurement noise structure for analyzing the affine class models, De Jong found the strong serial correlation in the measurement noise for one- and two-factor models[7], but did not provide a solution to this problem. Dempster and Tang examined the affine models for both bond yields and commodity futures and found that the main issue regarding measurement noise was serial correlation[8]. Then they took the commodity futures model as an example and proposed an augmented state space form containing "perfect" measurements, i. e., containing no noise, for the estimation of the model[8]. The purpose of this paper is to develop another augmented state space form containing sequentially correlated noise for an affine term structure model.

1 SYSTEM MODELS AND STATE SPACE MODELS

Vasicek model and cox-ingersoll-ross (CIR) model are the two most widely used affine ones[9-10]. In one-factor Vasicek model, the unob-

Received date: 2011-01-05; **revision received date:** 2011-03-10

E-mail: iamwushu@hotmail.com

servable state variable, that is, the instantaneous interest rate, follows Ornstein Unlenbeck process and therefore is assumed to revert to a long term rate. The main shortcoming of this model is that since the distribution of the state variable is normal, the model permits negative interest rates with a relatively low probability. CIR model uses a mean-reverting square-root process and preludes negative interest rates. Moreover, Vasicek is a Gaussian affine model and CIR model is a non-Gaussian one. A non-Gaussian system maybe loses certain accuracy in the linear Gaussian filtering process of Kalman filter[11]. So one-factor Vasicek model is chosen to demonstrate the developed strategy in this paper despite its possible negative outcome. What's more, from a pragmatic point of view, the less realistic model is much simpler to work with and eases some estimation difficulties.

1.1 System state space models

The Vasicek process satisfies the univariate stochastic differential equation

$$\mathrm{d}r(t) = k[\theta - r(t)]\mathrm{d}t + \sigma \mathrm{d}w(t) \tag{1}$$

When the short rate $r(t)$ deviates from its long term mean θ, the process will return to this mean at the mean-reverting intensity k. The term of $k[\theta - r(t)]\mathrm{d}t$ is the instantaneous drift, σ^2 the absolute volatility of the short rate , and $w(t)$ the standard Brownian motion. In terms of the state space form within Kalman filter framework, this equation is referred to as the transition equation. To apply Kalman filter, we need to derive the expressions for the conditional mean and variance of the unobservable state variable process over discrete time intervals. Considering the length of the paper, the details are not discussed. For the derivation, see Ref. [12]. The first two moments of the Gaussian transition density of $r(t)$ is

$$r(t+1)\,|\,r(t) \sim N\left(\theta(1 - \mathrm{e}^{-k\Delta t}) + \mathrm{e}^{-k\Delta t} r(t),\ \frac{\sigma^2}{2k}(1 - \mathrm{e}^{-2k\Delta t})\right) \tag{2}$$

where Δt is the time interval. Then the transition equation is specified as follows

$$r(t+1) = \theta(1 - \mathrm{e}^{-k\Delta t}) + \mathrm{e}^{-k\Delta t} r(t) + w(t) \tag{3}$$

where $w(t)$ is a random variable with zero mean and the variance given by $\left(\frac{\sigma^2}{2k}\right)(1 - \mathrm{e}^{-2k\Delta t})$.

As to measurement system, the term structure $\boldsymbol{z}(t,T)$ is

$$\boldsymbol{z}(t,T) = -\frac{\boldsymbol{A}(\tau)}{\tau} + \frac{\boldsymbol{B}(\tau)}{\tau} r(t) \tag{4}$$

where t, T and $\tau = T - t$ denote the current time, the contract maturity and the time to maturity respectively. The parameters $\boldsymbol{A}(\tau)$ and $\boldsymbol{B}(\tau)$ satisfy the following ordinary differential equations (ODEs)[9], that is

$$\boldsymbol{B}'(\tau) + k\boldsymbol{B}(\tau) = 1 \tag{5}$$

$$-\boldsymbol{A}'(\tau) - k\left(\theta - \frac{\sigma\lambda}{k}\right)\boldsymbol{B}(\tau) + \frac{\sigma^2}{2}\boldsymbol{B}^2(\tau) = 0 \tag{6}$$

where λ represents the market price of risk. ODEs can be solved by numerical integration with the boundary conditions: $\boldsymbol{B}(0) = 0_{m\times 1}$ with m referring to the number of state variables and $\boldsymbol{A}(0) = 0$. The measurement Eq. (4) is completed with

$$\begin{cases} \boldsymbol{B}(\tau) = \dfrac{1}{k}(1 - \mathrm{e}^{-k\tau}) \\ \boldsymbol{A}(\tau) = \dfrac{\gamma[\boldsymbol{B}(\tau) - \tau]}{k^2} - \dfrac{\sigma^2 \boldsymbol{B}^2(\tau)}{4k} \end{cases} \tag{7}$$

where $\gamma = k^2\left(\theta - \frac{\sigma\lambda}{k}\right) - \frac{\sigma^2}{2}$.

In the Vasicek model, the measurement equation represents the affine relationship between the zero coupon bond yields and the unobservable state variable. Following the assumption that the measurement noise in bond yields is sequentially correlated[8], the measurement equation in terms of state space form for observable yields is given by

$$\boldsymbol{z}(t+1) = -\frac{\boldsymbol{A}(\tau)}{\tau} + \frac{\boldsymbol{B}(\tau)}{\tau} r(t+1) + \boldsymbol{v}(t+1) \tag{8}$$

where the measurement noise is generated from

$$\boldsymbol{v}(t+1) = \boldsymbol{\Psi}\boldsymbol{v}(t) + \boldsymbol{u}(t+1) \tag{9}$$

where $\boldsymbol{\Psi}$ is a $n \times n$ matrix with n referring to the number of observations and $\boldsymbol{u}(t+1)$ the Gaussian random vector sequence (white noise) with zero mean and covariance matrix $\boldsymbol{R}$. In addition, $w(t)$ and $\boldsymbol{u}(t)$ are independent. The matrices $\boldsymbol{\Psi}$ and $\boldsymbol{R}$ are specified as

$$\boldsymbol{\Psi} = \begin{bmatrix} \Psi_1 & 0 & \cdots & 0 \\ 0 & \Psi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Psi_n \end{bmatrix} \tag{10}$$

$$\boldsymbol{R} = \begin{bmatrix} R_1^2 & 0 & \cdots & 0 \\ 0 & R_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n^2 \end{bmatrix} \tag{11}$$

Both Ψ_i and R_i are defined between -1 and 1 with $i=1,\cdots,n$.

1.2 System augmentation models

The state is first augmented to include $\boldsymbol{v}(t+1)$. Then the measurements are "perfect", i. e., containing no noise. The perfect measurements lead to singular problem within the framework of Kalman-Bucy theory[13]. To alleviate the problem in this case, we reconstruct the structure of the measurement noise by

$$\boldsymbol{V}(t+1) = \boldsymbol{v}(t+1) + \boldsymbol{\varepsilon}(t+1) \tag{12}$$

where $\boldsymbol{\varepsilon}(t+1)$ is a white noise with zero mean and covariance matrix $\boldsymbol{E}$

$$\boldsymbol{E} = \begin{bmatrix} E_1^2 & 0 & \cdots & 0 \\ 0 & E_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_n^2 \end{bmatrix} \tag{13}$$

where $-1 \leqslant E_i \leqslant 1$ with $i=1,\cdots,n$. The requirement that the covariance of $\boldsymbol{E}$ is small relative to $\boldsymbol{V}(t+1)$ gives the model little impact while producing filtered estimation. The state space form of Vasicek model after augmentation becomes

$$\boldsymbol{X}(t+1) = \boldsymbol{C} + \boldsymbol{\Phi}\boldsymbol{X}(t) + \boldsymbol{W}(t) \tag{14}$$

$$\boldsymbol{z}(t+1) = -\frac{\boldsymbol{A}}{\tau} + \boldsymbol{H}\boldsymbol{X}(t+1) + \boldsymbol{\varepsilon}(t+1) \tag{15}$$

where $\boldsymbol{X}(t+1) = [r(t+1) \quad \boldsymbol{v}(t+1)]^{\mathrm{T}}$, $\boldsymbol{C} = [\theta(1-\mathrm{e}^{-k\Delta t}) \quad \boldsymbol{0}_n]^{\mathrm{T}}$, $\boldsymbol{W}(t) = [w(t) \quad \boldsymbol{\varepsilon}(t)]^{\mathrm{T}}$, $\boldsymbol{H} = \begin{bmatrix} \dfrac{\boldsymbol{B}}{\tau} & \boldsymbol{I}_n \end{bmatrix}$, and the new covariance matrix for $\boldsymbol{W}(t)$ is $\boldsymbol{\Omega}$. The augmented matrices are

$$\boldsymbol{\Phi} = \begin{bmatrix} \mathrm{e}^{-k\Delta t} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Psi} \end{bmatrix} \tag{16}$$

$$\boldsymbol{\Omega} = \begin{bmatrix} \dfrac{\sigma^2}{2k}(1-\mathrm{e}^{-2k\Delta t}) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} \tag{17}$$

The method for optimal filtering developed by Kalman is applied to the augmented state space models as follows[14]

$$\begin{aligned} &\hat{\boldsymbol{X}}(t+1|t) = \boldsymbol{C} + \boldsymbol{\Phi}\hat{\boldsymbol{X}}(t) \\ &\boldsymbol{P}(t+1|t) = \boldsymbol{\Phi}\boldsymbol{P}(t|t)\boldsymbol{\Phi}^{\mathrm{T}} + \boldsymbol{\Omega} \\ &\hat{\boldsymbol{X}}(t+1|t+1) = \hat{\boldsymbol{X}}(t+1|t) + \boldsymbol{K}(t+1) \cdot \\ &\qquad \left[\boldsymbol{z}(t+1) + \frac{\boldsymbol{A}}{\tau} - \boldsymbol{H}\hat{\boldsymbol{X}}(t+1|t)\right] \\ &\boldsymbol{K}(t+1) = \boldsymbol{P}(t+1|t)\boldsymbol{H}^{\mathrm{T}} \cdot \\ &\qquad [\boldsymbol{H}\boldsymbol{P}(t+1|t)\boldsymbol{H}^{\mathrm{T}} + \boldsymbol{E}]^{-1} \\ &\boldsymbol{P}(t+1|t+1) = [\boldsymbol{I} - \boldsymbol{K}(t+1)\boldsymbol{H}] \cdot \\ &\qquad \boldsymbol{P}(t+1|t)[\boldsymbol{I} - \boldsymbol{K}(t+1)\boldsymbol{H}]^{\mathrm{T}} + \\ &\qquad \boldsymbol{K}(t+1)\boldsymbol{E}\boldsymbol{K}(t+1)^{\mathrm{T}} \end{aligned} \tag{18}$$

where $\hat{\boldsymbol{X}}(t+1|t+1)$ is the optimal estimate of $\boldsymbol{X}(t+1|t+1)$, $\boldsymbol{K}(t+1)$ the filter gain, and $\boldsymbol{P}(t+1|t+1)$ the covariance of the error between the actual $\boldsymbol{X}(t+1|t+1)$ and its estimate $\hat{\boldsymbol{X}}(t+1|t+1)$.

2 EXPERIMENTAL RESULTS

In this experiment, we use the developed augmented Kalman filter system to receive the bond yields as input. The monthly data used for the following example are taken from the website of Bank of Canada and the data series cover the period from January, 1991 to December, 2000, totalling four monthly yield series and 120 time series. The zero coupon bond yields incorporated into the proposed measurement system include four observations with 0.25-year, 0.50-year, 1.00-year, and 5.00-year time to maturities.

The performance criterion used for evaluating the proposed augmented Kalman filter algorithm is the residuals of the model, defined by the difference between the observable bond yields and the filtered yields. The residual results from the correlated noise assumption are compared with the results from the i. i. d assumption. Tables 1 and 2 provide the mean and standard deviation of the residuals with both assumptions. The mean is calculated as the time series average of residuals for four observations respectively and it should be close to zero. Generally speaking, the model slightly underestimates the bond yields. The model with correlated noise assumption provides a substantially better fit of the term structure data than the one with i. i. d assumption. The mean and standard deviation in the former are smaller than those in the latter.

Table 1 Residuals of Vasicek model with correlated noise assumption

Maturity	0.25-year	0.50-year	1.00-year	5.00-year
Mean	−0.000 102	0.000 215	0.000 697	0.000 902
Standard deviation	0.000 174	0.000 294	0.000 302	0.000 611

Table 2 Residuals of Vasicek model with i. i. d assumption

Maturity	0.25-year	0.50-year	1.00-year	5.00-year
Mean	0.001 298	0.004 866	0.005 490	0.008 116
Standard deviation	0.001 548	0.001 469	0.001 461	0.001 543

The comparison results of the model with different measurement noise structures are illustrated in Figs. 1-4, which graph the filtered term structure and the observable term structure for four observations respectively . In Figs . 1 - 4 , "Ture" represents the observable term structure, "Estimate/white noise" the filtered term structure with white noise assumption and "Estimate/correlated noise" the filtered term structure with correlated noise structure. The model fits well on the short end of the yield curve but a little bit poorly on the long end. The accuracy of this technique appears to somewhat deteriorate with the fourth observation. This should not be too surprising and this is probably because we use a one-factor model. According to Ref. [7], with more factors adding to the model, the model will give more adequate fitting result. For most one-factor model, the only factor is always considered to be the instantaneous short rate[15-16]. Usually, in a three-factor term structure model, the second factor is the stochastic long-run mean of the short rate and the third one is the spread factor which represents the spread between rates of different maturities, especially the difference between the short end and the long end[16]. Deductively, with adding the second and third factors, more accurate filtering result, particularly that of the long end, can be achieved. As a result, although they are slightly less encouraging for the fourth observation, the results are on the whole illustrative of the point that the model with correlated noise assumption provides a substantially better filtering effect than the model with white noise assumption. Based on the above analysis, we can conclude that the correlated noise assumption is more reasonable and the proposed augmented Kalman filter is a successful technique for determining the unobservable state variable.

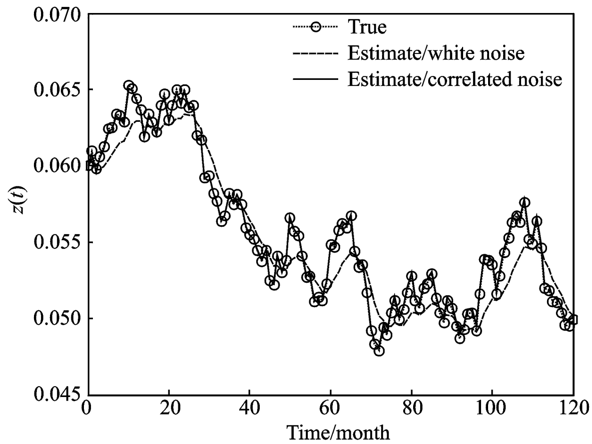

Fig. 1 Comparison results for 0.25-year

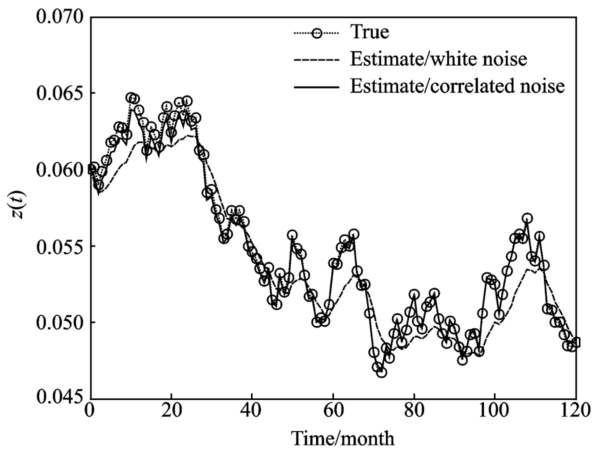

Fig. 2 Comparison results for 0.50-year

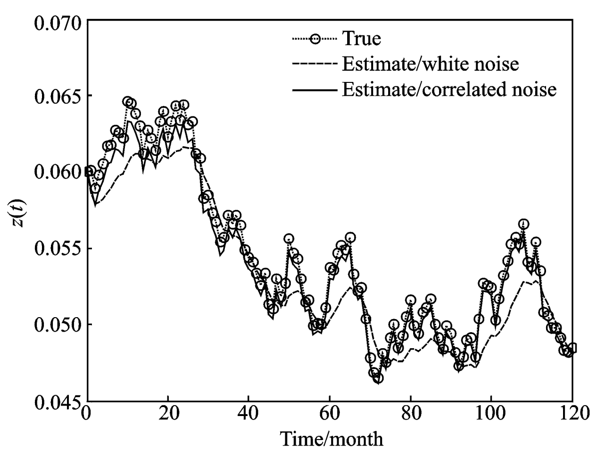

Fig. 3 Comparison results for 1.00-year

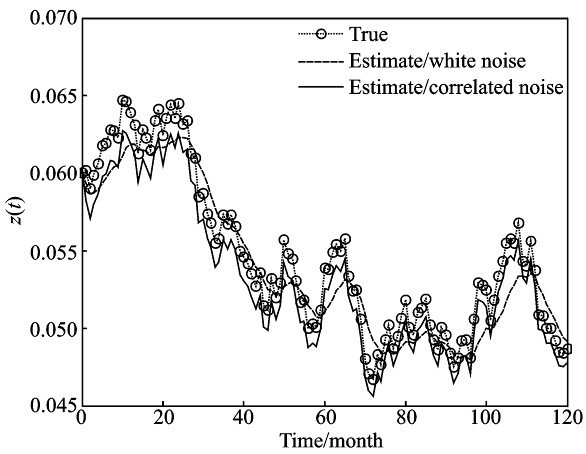

Fig. 4 Comparison results for 5.00-years

3 CONCLUSION

The goal of this paper is to consider the practical details in estimating the unobservable variable for the one-factor Vasicek term structure model. Following the assumption of sequentially correlated measurement noise, the term structure estimation technique based on augmented Kalman filter is developed, which overcomes the singular problem by adding a small white measurement noise. The experimental results illustrate the performance of the developed filter technique for the model estimation and show that the model with sequentially correlated noise assumption works much better than the model with i. i. d assumption in estimating the unobservable state variable for Vasicek term structure model.

References:

[1] Song Futie, Chen Langnan. Kalman filter approach to simulate and estimate the term structure of interest rates in Shanghai stock exchange [J]. Management Science, 2006, 19(6): 81-88. (in Chinese)

[2] Sanforda A D, Martin G M. Simulation-based Bayesian estimation of an affine term structure model [J]. Computational Statistics and Data Analysis, 2005, 49(2): 527-554.

[3] Cheridito P, Filipović D, Kimmel R L. A note on the Dai-Singleton canonical representation of affine term structure models [J]. Mathematical Finance, 2010, 20(3): 509-519.

[4] Cortazar G, Schwartz E S, Naranjo L. Term structure estimation in low-frequency transaction markets: a Kalman filter approach with incomplete panel-data [EB/OL]. http://papers.ssrn.com/sol3/papers.cfm?abstract_id=567090, 2004-07/2010-12.

[5] Duan J C, Simonato J G. Estimating and testing exponential-affine term structure models by Kalman filter [J]. Review of Quantitative Finance and Accounting, 1999, 13(2): 111-135.

[6] Christensen J H E, Diebold F X, Rudebusch G D. An arbitrage free generalized Nelson Siegel term structure model [J]. Econometrics Journal, 2009, 12(3): C33-C64.

[7] De Jong F. Time series and cross-section information in affine term structure models [J]. Journal of Business & Economic Statistics, 2000, 18(3): 300-314.

[8] Dempster M A H, Tang K. Estimating exponential affine models with correlated measurement errors: applications to fixed income and commodities [J]. Journal of Banking & Finance, 2011, 35(3): 639-652.

[9] Vasicek O. An equilibrium characterization of the term structure [J]. Journal of Financial Economics, 1977, 5(2): 177-188.

[10] Cox J C, Jr Ingersoll J E, Ross S A. An intertemporal general equilibrium model of asset prices [J]. Econometrica, 1985, 53(2): 363-384.

[11] Hua Bing, Liu Jianye, Li Rongbing, et al. Redundant MEMS-IMU/GPS integrated navigation system based on improved unscented particle filter algorithm [J]. Journal of Nanjing University of Aeronautics & Astronautics, 2007, 39(5): 570-575. (in Chinese)

[12] Wang Anxing. Interest rate models [M]. Shanghai: Press of Shanghai University of Finance and Economics, 2007: 83-84. (in Chinese)

[13] Kalman R E, Bucy R S. New results in linear filtering and prediction theory [J]. Transactions of the ASME, Series D, Journal of Basic Engineerning, 1961, 83:95-107.

[14] Kalman R E. New methods in Wiener filtering theory [C]//Proceedings of the First Symposium on Engineering Applications of Random Function Theory and Probability. New York: Wiley, 1963: 270-388.

[15] Theoret R, Rostan P, El Moussadek A. Forecasting the interest rate term structure: using the model of Fong & Vasicek, the extended Kalman filter and the Bollinger bands [EB/OL]. http://papers.ssrn.com/sol3/papers.cfm?abstract_id = 671581, 2005-

03/2010-12.

[16] Gao Chi, Wang Qing. Term structure of interest rate in China with Kalman filter [J]. Southern Economy, 2006 (12): 19-26. (in Chinese)

带有序列相关噪声Vasicek期限结构模型的线性滤波

吴 姝 刘思峰

(南京航空航天大学经济与管理学院，南京，210016，中国)

摘要：在基于Vasicek 利率期限结构模型，使用Kalman 滤波器进行状态估计时，通常采用量测噪声不相关假设。研究表明，量测噪声在很多情况下满足相关性假设。本文采用一般性的相关量测噪声假设，提出针对Vasicek 模型潜变量估计的状态扩维Kalman 滤波法。实验结果表明，量测噪声相关假设下的Vasicek 模型比较准确地描述了利率期限结构的动态变化特征。

关键词：Vasicek 期限结构模型；扩维卡尔曼滤波；序列相关噪声；状态估计

中图分类号：F830.9

(Executive editor: Zhang Huangqun)