MULTIGRID TECHNIQUE APPLIED TO LINEAR TIME-DEPENDENT HYPERBOLIC SYSTEMS

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Abstract: A system of linear time-dependent hyperbolic partial differential equations in the form of the time-domain Maxwell's equations is numerically solved using a geometric multigrid method. The multilevel method is an adaptation of Ni's cell-vertex based multigrid technique, originally proposed for accelerating steady state convergence of nonlinear time-dependent Euler equations of gas dynamics. We discuss issues pertaining to the application of the geometric multigrid method to a system of equations where the major issue is of accurately propagating linear waves over large distances leading to major constraints on the required grid resolution in terms of points-per-wavelength.

Key words: multigrid; linear; hyperbolic; finite volume; maxwell

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INTRODUCTION

Multigrid methods in the geometric form have been proven to be very efficient in accelerating convergence of linear boundary value problems by using a solution space consisting of a hierarchy of meshes from fine to coarse. They take the advantage of efficient smoothing high frequency error components in the iterative solution of the system of linear algebraic equations arising from discretization of linear elliptic partial differential equations (PDEs). The multigrid technique has, in the past, also been extended to accelerate solution of nonlinear hyperbolic time-dependent PDEs in the form of the time-dependent, inviscid Euler equations of gas dynamics, where solutions are usually severely constrained due to time-step restrictions arising out of stability considerations. The arguments for multigrid efficiency in this case are usually heuristic in nature, and are based on the perceived ability of coarser grids, subjected to more relaxed stability criteria, being able to rapidly expel disturbances from the computational domain. The cell-vertex multigrid technique due to Ni[1] is a good example of such a multigrid technique where a hierarchy of discretizations is used to obtain rapid convergence of the Euler equations of gas dynamics while also provide a cell vertex framework for the finite volume discretization. The multigrid technique, in general, has not been commonly applied to linear time-dependent hyperbolic systems like the time-domain Maxwell's equations where along with stability considerations, strict limitations are placed on the discretization in terms of points-per-wavelength to accurately propagate linear waves over long distances. In the present work, the geometric multigrid method is used to solve the time-domain Maxwell's equations in a finite volume time domain (FVTD) framework. The multilevel method used here is an adaptation of Hall's[2] extension of Ni's cell-vertex based multigrid technique, originally proposed to accelerate the convergence of the time-dependent Euler equations of gas dynamics. FVTD methods are increasingly employed to compute electromagnetic scattering because of their greater flexibility in dealing with broad-band signals and diverse material proper-

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ties[3-4]. However, FVTD methods are of limited applicability for practical applications involving large electric sizes. The computational grid for FVTD methods can be based on resolution of 10—20 points per wavelength (PPW) resulting in very fine meshes at large electric sizes along with very small time step. In addition, computations are carried out for many sinusoidal time cycles to achieve harmonic steady state. This results in long computational times for FVTD techniques make many common engineering applications prohibitively expensive. In the current multilevel application, the time-domain Maxwell equations are solved to a harmonic steady state on a hierarchy of meshes using Ni′s approach. The linear nature of Maxwell′s equations allows for a more accurate representation of the fine-grid problem on the coarse grid due to the constant Jacobian matrix. Artificial viscosity is also not required to smoothen interpolation errors as in the nonlinear case. The major drawback, as compared to other well known multigrid applications, is the need for maintaining a resolution of at least 5—6 PPW on the coarsest level, for accurately simulating wave propagation both in terms of phase and amplitude on the coarse grid to limit the maximum number of levels that can be traversed in a multigrid cycle.

1 GOVERNING EQUATIONS AND NUMERICAL METHOD

Maxwell′s equations for electromagnetic wave propagation in free space, can be expressed in 2-D conservative form for transverse magnetic (TM) or transverse electric (TE) waves, simply as

$$\frac{\partial \boldsymbol{u}}{\partial t}+\frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x}+\frac{\partial \boldsymbol{g}(\boldsymbol{u})}{\partial y}=0 \tag{1}$$

The FVTD technique numerically solves the integral form of Eq. (1) in a discrete finite volume framework and is described in detail in Refs. [3-5]. In the scattered field formulation used here, the scattered field variables are solved and an analytically defined incident field is assumed to be available.

1.1 Ni′s multigrid method and Hall′s extension

Ni′s basic cell-vertex based finite volume scheme can be considered to belong to the fluctuation-splitting framework for the solution of hyperbolic conservation laws. In Ni′s finite-volume time integration scheme, the fluctuation is calculated based on state vector stored at cell vertices and distributed to cell vertices after a discrete time-step. This distribution finally leads to second-order accurate cell-vertex based Lax-Wendroff scheme[1]. Ni, in his original paper, required heavy numerical damping while solving for strongly nonlinear systems encountered in the form of transonic flows in gas dynamics. The linear nature of the time-dependent Maxwell′s equation in free space may be a more appropriate choice for the application of Ni′s novel cell-vertex based finite volume scheme, as there is no reliance on user defined numerical damping to stabilize the scheme[5]. This basic solution technique starts with the calculation of ″change″ in a control volume based on cell vertex flux values and forms the first-order term in the Lax-Wendroff correction. This first-order discrete numerical ″change″ $\Delta\boldsymbol{U}$ for an arbitrary quadrilateral cell c is approximated using the divergence theorem as

$$\Delta\boldsymbol{U}_c=\frac{\Delta t}{\Delta A_c}\Big(\sum_{p=1}^{4}[(\boldsymbol{F}(\boldsymbol{u})\boldsymbol{n}_x+\boldsymbol{G}(\boldsymbol{u})\boldsymbol{n}_y)s]_p\Big) \tag{2}$$

where ΔA_c is the area of the cell c, s the face length with an outer unit normal vector $\boldsymbol{n}$, the Cartesian components of which are $\boldsymbol{n}_x$ and $\boldsymbol{n}_y$ and Δt the time step restricted by the Courant-Friedrich-Lewy (CFL) stability criteria, flux vectors $\boldsymbol{F}(\boldsymbol{u})$, $\boldsymbol{G}(\boldsymbol{u})$ are computed for each pth cell face by taking average of the flux vectors stored at vertices of the face (Fig. 1).

The first-order change in fine grid at grid point 1 is obtained as an area weighted quantity

$$(\Delta\boldsymbol{U}_h^n)=\frac{\Delta\boldsymbol{U}_a\Delta A_a+\Delta\boldsymbol{U}_b\Delta A_b+\Delta\boldsymbol{U}_c\Delta A_c+\Delta\boldsymbol{U}_d\Delta A_d}{\Delta A_a+\Delta A_b+\Delta A_c+\Delta A_d} \tag{3}$$

This change is used to determine the second-order contribution in the total Lax-Wendroff correction to be distributed to relevant cell vertices in order to update the state vector[1-2]. The total correc-

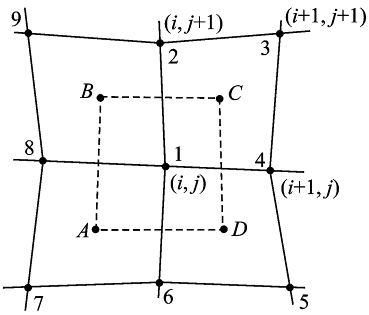

Fig. 1 2-D arbitrary computational cell

tion at grid point 1, $\delta \boldsymbol{U}_h$, is obtained by adding the first-order and the second-order contributions as

$$(\delta \boldsymbol{U}_h)_1 = \Delta \boldsymbol{U}_1^n + [(\Delta \boldsymbol{F}_b - \Delta \boldsymbol{F}_d)\Delta y_2 + (\Delta \boldsymbol{F}_a - \Delta \boldsymbol{F}_c)\Delta y_1 + (\Delta \boldsymbol{G}_d - \Delta \boldsymbol{G}_b)\Delta x_2 + (\Delta \boldsymbol{G}_c - \Delta \boldsymbol{G}_a)\Delta x_1]\frac{\Delta t}{4\Delta A_1} \tag{4}$$

where $\Delta x_1 = (x_b - x_d)$, $\Delta x_2 = (x_c - x_a)$, $\Delta y_1 = (y_b - y_d)$ and $\Delta y_2 = (y_c - y_a)$. The state vector at next time level is updated by adding the total correction $\delta \boldsymbol{U}_h$ to the state vector at time level n. Unsteady fluxes $\Delta \boldsymbol{F}_c$ and $\Delta \boldsymbol{G}_c$ in Cartesian x and y directions respectively for cell c are found by evaluating the Jacobians at the corresponding vertices as[2]

$$\Delta \boldsymbol{F}_c = \frac{1}{4}\left[\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right)_1 + \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right)_2 + \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right)_3 + \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right)_4\right]\Delta \boldsymbol{U}_c \tag{5}$$

$$\Delta \boldsymbol{G}_c = \frac{1}{4}\left[\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\right)_1 + \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\right)_2 + \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\right)_3 + \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\right)_4\right]\Delta \boldsymbol{U}_c \tag{6}$$

The multigrid technique interwoven with the basic time integration technique employs progressively coarser grids to propagate the fine grid correction rapidly in the computational domain. When this time-stepping procedure is applied on the coarse levels, the change or first-order correction for the coarse mesh cells is estimated as the weighted average of the total corrections of the fine mesh nodes is defining the corresponding coarse mesh cell. Second-order correction terms to fine-grid accuracy on coarse meshes are then calculated based on this first-order correction and the Jacobian matrix. This is then distributed to the corners of the coarse mesh cells by the same distribution formula. This procedure is repeated for several coarse levels. Once the coarsest grid is reached, the coarser grid correction is then interpolated back on the fine grid and added to the solution of the fine grid. The multigrid process is shown schematically in Fig. 2 and discussed in detail in Ref. [6]. In Fig. 2, I represents an interpolation operator and Ω_h a grid. It may be noted that second-order terms in the coarse grid can be described purely in terms of the first-order changes in linear systems since the Jacobian matrix requirement for defining unsteady flux terms is constant. This is in contrast to that for nonlinear system where the Jacobian matrix on the coarse grids requires to be approximated using fine grid values.

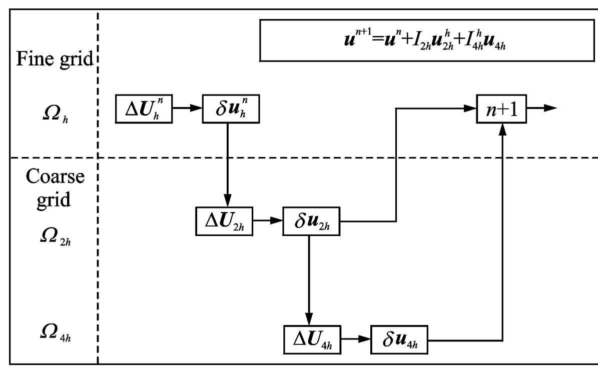

Fig. 2 Schematic of multigrid process

2 SAMPLE RESULTS

Multigrid results using two and three levels are compared with single (fine) grid solutions. Sample results are presented for electromagnetic scattering from perfectly conducting (PEC) circular cylinder with $a/\lambda = 9.6$ and $a/\lambda = 14.4$, where a is the radius of cylinder and λ the wavelength of the incident harmonic TM wave. More details and results can be found in Ref. [6]. An 'O' topology grid is used for the discretization with coarse meshes obtained by amalgamating four constituent fine-grid cells. Computations are carried out for the circular cylinder using two-level and three-level multigrid scheme with grid resolution of 22 PPW and 26.7 PPW respectively at the finest level on the scatterer surface. Figs. 3－4 show a comparison of two-level multigrid ($a/\lambda = 14.4$) and three-level multigrid ($a/\lambda = 9.6$) solution

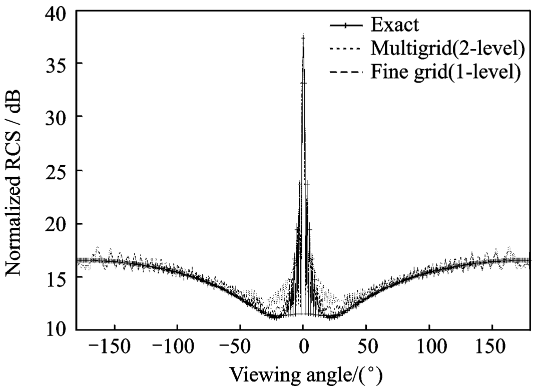

Fig. 3 Bistatic RCS, circular cylinder (2-level, $a/\lambda=14.4$)

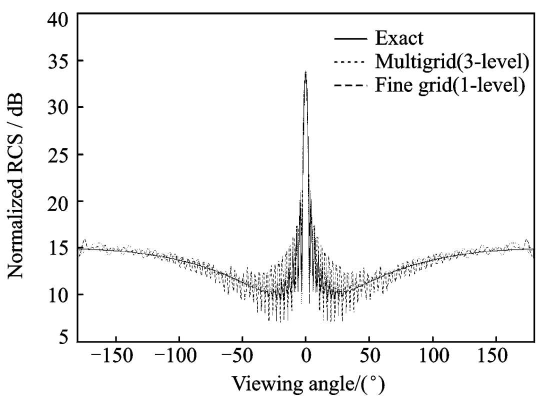

Fig. 4 Bistatic RCS, circular cylinder (3-level, $a/\lambda=9.6$)

with fine grid solution and the exact result, where RCS means radar cross section. Almost fine-grid accuracy is obtained except for a narrow region in the shadow area between ±50°. Bistatic RCS obtained using three-level multigrid method is further compared with a single-level solution on purely the coarsest grid with resolution of 6.67 PPW on scatterer surface. This comparison is done to bring out the ability of the present multigrid method to enforce almost fine-grid accuracy while cycling the solution through a hierarchy of grids. This comparison in Fig. 5 shows the enhanced accuracy of the three-level multigrid solution when compared to a solution obtained on purely the coarsest grid. The coarse grid solution deviates from the exact solution even at the important monostatic point ±180°. Numerical experiments further indicate a minimum of at least 5—6 PPW on the coarsest mesh to accurately simulate wave propagation which limits the number of coarser levels that can be traversed in such applications.

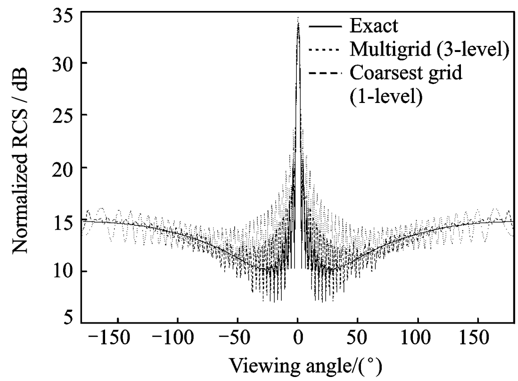

Fig. 5 Bistatic RCS, circular cylinder (3-level and coarsest grid, $a/\lambda=9.6$)

Sample results are also presented for electromagnetic scattering from NACA0012 subjected to broadside incidence, with $a/\lambda=10.0$, where a is the airfoil chord length and λ the wavelength of the incident harmonic TM wave. Computations are carried out for an airfoil using a three-level multigrid scheme with grid resolution of 24.7 PPW at the finest level on the scatterer surface. Fig. 6 shows a comparison of bistatic RCS obtained using three-level multigrid and fine grid solution with the Ref. [3] results. Almost fine-grid accuracy is obtained using the three-level multigrid scheme.

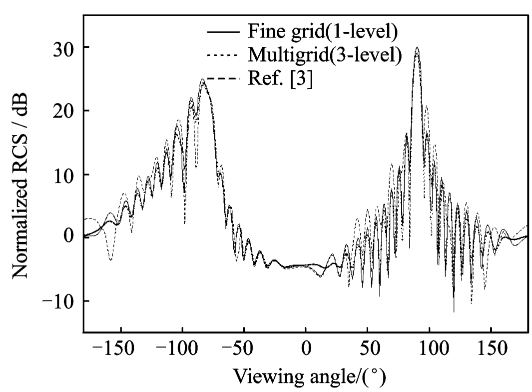

Fig. 6 Bistatic RCS, NACA 0012 (3-level and coarsest grid, $a/\lambda=10.0$)

3 CONCLUSION

The geometric multigrid method is used to achieve a faster convergence to harmonic steady state in the numerical solution of time-domain Maxwell equations for electromagnetic scattering problems using the FVTD technique. Unlike more common multigrid application involving linear boundary value problems, the major issue in the numerical solution of linear time-dependent hyperbolic PDEs is the accurate propagation of linear waves over large distances. In the current application using Hall′s modification of the Ni′s multigrid method, the linear Maxwell′s system does not requires the Jacobian to be approximated on coarser levels leading to a more accurate implementation. No artificial damping is required to stabilize the multigrid technique for the linear Maxwell′s system or to dampen interpolation errors. But a minimum PPW of 5—6 on the coarsest level for accurate wave propagation places major limitations on the number of levels that can be traversed in the multigrid framework leading to a modest 30%—40% reduction in overall CPU time.

References:

[1] Ni R-H. A multiple-grid scheme for solving the Euler equations[J]. AIAA Journal, 1982,20(11): 1565-1571.

[2] Hall M G. Numerical methods for fluid dynamics II: Cell-vertex multigrid schemes for solution of the Euler equations[M]. Oxford: Clarendon Press, 1985: 303-345.

[3] Shankar V, Hall W F, Mohammadin A H. A time-domain differential solver for electromagnetic scattering problems[J]. Proceedings of the IEEE, 1989,77(5): 709-721.

[4] Chatterjee A, Shrimal A. Essentially nonoscillatory finite volume scheme for electromagnetic scattering by thin dielectric coatings[J]. AIAA Journal, 2004, 42(2): 361-365.

[5] Deore N, Chatterjee A. A cell-vertex finite volume time domain method for electromagnetic scattering[J]. Progress in Electromagnetics Research M, 2010,12: 1-15.

[6] Deore N, Chatterjee A. Cell-vertex based multigrid solution of the time-domain Maxwell′s equations[J]. Progress in Electromagnetics Research B, 2010, 23: 181-197.

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