LONG-TERM RIGOROUS NUMERICAL INTEGRATION OF NAVIER-STOKES EQUATION BY NEWTON-GMRES ITERATION

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Abstract: The recent result of an orbit continuation algorithm has provided a rigorous method for long-term numerical integration of an orbit on the unstable manifold of a periodic solution. This algorithm is matrix-free and employs a combination of the Newton-Raphson method and the Krylov subspace method. Moreover, the algorithm adopts a multiple shooting method to address the problem of orbital instability due to long-term numerical integration. The algorithm is described through computing the extension of unstable manifold of a recomputed Nagata's lower-branch steady solution of plane Couette flow, which is an example of an exact coherent state that has recently been studied in subcritical transition to turbulence.

Key words: long-term numerical integration; Newton-Raphson iteration; general minimal residual (GMRES); multiple shooting; unstable manifold

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INTRODUCTION

A dynamical system theory has recently turned out to be useful to elucidate subcritical transition to turbulence by studying exact coherent states, i. e., equilibria and periodic solutions. In shear flow, an invariant set in state space that has only one unstable direction in phase space is called an "edge state"[1].

The edge state has a special property in which its stable manifold separates the laminar and turbulent basins. State points within the laminar-turbulent boundary are attracted to the edge state. For initial conditions just exceeding a critical value, corresponding state points will escape out of the laminar basin along the unstable manifold of the edge state.

Extension of unstable manifold encounters orbital instability for long-term numerical integration. To minimize instability, a multiple shooting method[2] is applied. The basic idea of this multiple shooting method is to cut the piece of orbit to be extended into several segments, solve a boundary value problem for each, and then concatenate the segments by a "gluing condition". Cutting the orbit into several segments reduces the propagation of numerical error through time by shortening the time interval.

The introduction of Krylov subspace method that is known as general minimal residual (GMRES)[3] in Newton-Raphson has taken a major step in solving linear equations. This combination of Newton-Raphson and GMRES is often referred to as Newton-Krylov iteration. This method has recently been used extensively in fluid dynamics to solve large system of ordinary linear equations arising from the discretization of the Navier-Stokes equation.

In this paper, an orbit continuation algo-

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rithm that uses Newton-Krylov iteration is illustrated to solve a system of linear equations produced by multiple shooting method. This algorithm is used to extend the unstable manifold of a recomputed Nagata′s lower-branch steady solution[4]. In minimal Couette flow[5], the solution appears for an elongated streamwise length of the computational box[6].

1 NEWTON-KRYLOV COMPUTATION OF STEADY SOLUTION

The plane Couette flow, where the Reynolds number is based on half the difference of the wall velocities U and half the separation of walls h, is studied. The computation is performed on a computational box with streamwise period $L_x = 1.963\pi h$ and spanwise period $L_z = 1.2\pi h$. The walls are separated by a distance $2h$. A resolution of $32\times 33\times 32$ number of grid points is used in the streamwise, wall-normal, and spanwise directions, respectively. The dealiased Fourier expansions are employed on the wall-parallel directions and Chebyshev-polynomial expansion on the wall-normal direction. The number of degrees of freedom of the discretized system in Navier-Stokes equation is $N = 11\ 117$. The Nagata′s lower-branch steady solution is obtained using Newton-Krylov iteration. This steady solution has only one unstable direction in phase space and thus an example of a simple edge state.

2 ORBIT CONTINUATION BY TIME INTEGRATION

Consider an N-dimensional dynamical system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{x}\in \mathbf{R}^N$ represents a state point in an N-dimensional state space and $\boldsymbol{f}\in \mathbf{R}^N$ represents the vector field obtained from the Navier-Stokes equation. The orbit $\boldsymbol{x}(t)$ is given by the time integration of Eq. (1) at any time t. The initial condition is fixed as

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 + \varepsilon \boldsymbol{v}_0 \tag{2}$$

where $\boldsymbol{x}_0$ is the recomputed Nagata′s lower-branch steady solution, $\boldsymbol{v}_0 \in \mathbf{R}^N$ the unstable eigenvector of the steady solution, and ε a small parameter. Eq. (1) is integrated until it intersects a fixed plane

$$g(\boldsymbol{x}(T)) = c \tag{3}$$

where g is a scalar function and c is a constant. The choice of g may depend on the problem at hand. In this study g refers to the sum of energy input (I) and dissipation (D) rates, both normalized with respect to their laminar state values, given as

$$I = \frac{1}{2L_xL_z/h}\int_0^{L_x}\int_0^{L_z}\left(\left.\frac{\partial u}{\partial y}\right|_{y=-h} + \left.\frac{\partial u}{\partial y}\right|_{y=+h}\right)\mathrm{d}x\mathrm{d}z \tag{4}$$

$$D = \frac{1}{2L_xL_zU^2/h}\int_0^{L_x}\int_{-h}^{+h}\int_0^{L_z} |\boldsymbol{\omega}|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{5}$$

where u in Eq. (4) is the streamwise velocity and $\boldsymbol{\omega}$ in Eq. (5) the vorticity vector. Other choices of g include fixing the integration time or length of the segment. From here on we call this orbit continuation by time integration as single shooting.

3 ORBIT CONTINUATION BY MULTIPLE SHOOTING METHOD

Consider the simplest example of multiple shooting method in which the piece of orbit is to be cut into two segments. Let the two segments be N-dimensional column vectors $\boldsymbol{\gamma}_1(t_1)$ and $\boldsymbol{\gamma}_2(t_2)$ with integration times T_1 and T_2, respectively. The initial condition of $\boldsymbol{\gamma}_1(t_1)$ is fixed as

$$\boldsymbol{\gamma}_1(0) = \boldsymbol{x}_0 + \varepsilon \boldsymbol{v}_0 \tag{6}$$

which is essentially the same as that of Eq. (2). Using Eq. (6) as an initial condition, Eq. (1) is integrated until it intersects a fixed plane

$$g(\boldsymbol{\gamma}_1(T_1)) = c_1 \tag{7}$$

for a constant value c_1. When Eq. (7) is achieved, T_1 and the initial point of the second segment $\boldsymbol{\gamma}_2(0)$ are made fixed, and integration for the second segment is resumed until the orbit intersects a final plane

$$g(\boldsymbol{\gamma}_2(T_2)) = c_2 \tag{8}$$

for a constant value c_2. T_2 is made fixed after the second integration. The segments are ensured continuous by the gluing condition

$$\boldsymbol{\gamma}_2(0) - \boldsymbol{\gamma}_1(T_1) = 0 \tag{9}$$

The $(N+2)$ Eqs. (7–9) are solved simultaneously for $(N+2)$ unknowns $\boldsymbol{\gamma}_2(0)$, T_1 and T_2 using Newton-Krylov iteration.

For triple shooting there will be an increase of $(N+1)$ unknowns. These unknowns are the initial point of the third segment $\boldsymbol{\gamma}_3(0)$ and its integration time T_3. Also, $(N+1)$ equations will be added to the system of linear equations. These equations are the new final plane, constant under c_3, for the termination of the integration for the third segment and the gluing condition for the second and third segments, given respectively as

$$g(\boldsymbol{\gamma}_3(T_3)) = c_3 \tag{10}$$

$$\boldsymbol{\gamma}_3(0) - \boldsymbol{\gamma}_2(T_2) = 0 \tag{11}$$

The $(2N+3)$ equations for triple shooting are solved simultaneously. For every increase in the number of segments there will be a corresponding increase of $(N+1)$ unknowns coming from the initial point of the new segment and its integration time, as well as increase of $(N+1)$ equations coming from the new final plane and gluing condition. To sum up: For i segments, there are $((i-1)N+i)$ equations to be solved simultaneously using Newton-Krylov iteration for $((i-1)N+i)$ unknowns.

4 RESULTS AND DISCUSSION

Orbit continuation is performed on the one-dimensional unstable manifold of a recomputed Nagata's lower-branch steady solution by single shooting as well as double and triple shooting methods for a fixed value of ε of the initial condition. Fig. 1 is a piece of the extended unstable manifold of the steady solution projected onto I-D plane. The dots denote the instantaneous points on the extended unstable manifold where visualization of flow structure is taken.

The unstable manifold stays for some time around the energy level of the steady solution and then rapidly escapes from that state. The difference of the state points in the final plane of two different types of shooting is calculated, that is

$$\text{difference} = \frac{\| \boldsymbol{X}' - \boldsymbol{X} \|}{\| \boldsymbol{X}' \|} \tag{12}$$

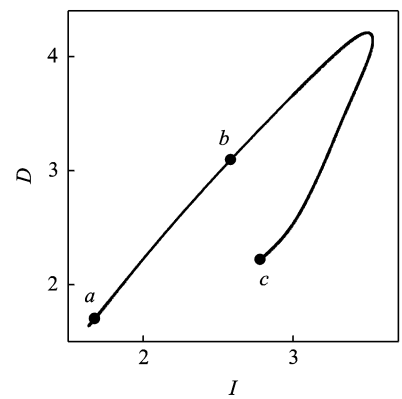

Fig. 1 A piece of extended unstable manifold of recomputed Nagata's lower-branch steady solution

where $\boldsymbol{X}$ and $\boldsymbol{X}'$ are state points in the final plane, the latter is from the shooting with more segments, and $\| \cdot \|$ denotes the Euclidian norm. It is found that until before the continued unstable manifold makes a return trip to the steady solution, the state point in the final plane of either double or triple shooting is comparable with that of single shooting, at least for 10^{-8}. However, during the trip back to the steady solution, the result of single shooting deviates significantly from that of double or triple shooting method. The difference between the state points in the final plane of double or triple shooting has increased as much as 10^{-3} in comparison to that of single shooting. On the other hand, the state points in the final plane by use of double and triple shootings remain consistent with each other even for continued orbit returning back to the steady solution, at least for 10^{-8}.

For double shooting the choice of the first plane does not affect the result of the final plane. The same is true for triple shooting in the case of its first and second planes. Hence, we are free from setting a particular choice of middle plane(s).

The behavior of the flow structure during the extension of the unstable manifold is observed. Figs. 2–4 are the flow structures at points a, b, and c on the unstable manifold in Fig. 1, respectively. The gray isosurface is the null streamwise velocity, in which the corrugation represents streamwise streaks. The red and blue isosurfaces, which is at $1.0(U/h)^2$ of the second invariant of velocity gradient tensor, represent vortex tubes for clockwise and counter-clockwise streamwise vorticity components, respectively.

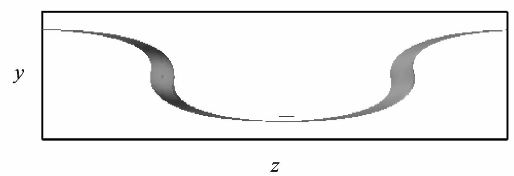

Fig. 2 Flow structure at point *a* in Fig. 1

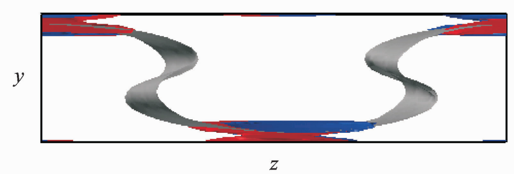

Fig. 3 Flow structure at point *b* in Fig. 1

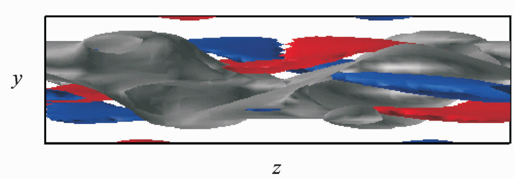

Fig. 4 Flow structure at point *c* in Fig. 1

At point *b*, patches of streamwise vortices start to develop at the crest and valley of the streak. Also, the shape of the streak is starting to be deformed from that at point *a*. This deformation is due to the spanwise bending of the streak induced by the instability. At point *c*, the patches of streamwise vortices have grown in size and have completely deformed the shape of the streak. These streamwise vortices are known to be necessary structures found in the regeneration cycle in near-wall turbulence[5].

5 CONCLUSION

An orbit continuation algorithm is described, which adopts a multiple shooting method by extending the unstable manifold of a recomputed Nagata′s lower-branch steady solution. For short orbit, the result of single shooting is consistent with that of multiple shooting. However, for longer orbit, single shooting suffers numerical error through time. This error is reduced by shortening the integration time through the application of multiple shooting. The flow structures at various points on the extended unstable manifold are observed, where the presence of streamwise vortices which are necessary structures in near-wall turbulence is found.

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