Fractional Pfaff-Birkhoff Principle and Birkhoff′s Equations in Terms of Riesz Fractional Derivatives

Zhou Yan(周燕)[1], *Zhang Yi*(张毅)[2*]

1. College of Mathematics and Physics, Suzhou University of Science and Technology, Suzhou, 215009, P. R. China;
2. College of Civil Engineering, Suzhou University of Science and Technology, Suzhou, 215009, P. R. China

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Abstract: The dynamical and physical behavior of a complex system can be more accurately described by using the fractional model. With the successful use of fractional calculus in many areas of science and engineering, it is necessary to extend the classical theories and methods of analytical mechanics to the fractional dynamic system. Birkhoffian mechanics is a natural generalization of Hamiltonian mechanics, and its core is the Pfaff-Birkhoff principle and Birkhoff′s equations. The study on the Birkhoffian mechanics is an important developmental direction of modern analytical mechanics. Here, the fractional Pfaff-Birkhoff variational problem is presented and studied. The definitions of fractional derivatives, the formulae for integration by parts and some other preliminaries are firstly given. Secondly, the fractional Pfaff-Birkhoff principle and the fractional Birkhoff′s equations in terms of Riesz-Riemann-Liouville fractional derivatives and Riesz-Caputo fractional derivatives are presented respectively. Finally, an example is given to illustrate the application of the results.

Key words: fractional derivative; fractional Pfaff-Birkhoff principle; fractional Birkhoff′s equation; transversality condition

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1 Introduction

Fractional calculus first appearing in the letter that L′Hospital wrote to Leibniz asking about the nth-derivative of a function in 1695. If $n=1/2$, what would the result be. Although the basic mathematical ideas were explored long ago by the mathematicians Euler, Laplace and Fourier, the development of the fractional calculus is so slow that it was not until 1974 that the first book on this topic was published[1].

Recent decades have seen the wide application of factional calculus in many fields such as physics, chemistry, biology, electronics, economics, and control systems[2-3]. In 1996 and 1997, Riewe[4-5] published the papers about his work on fractional variational problems , in which the fractional calculus is applied to a nonconservative mechanical system. Since then, the fractional variational problems have been studied by many researchers. Agrawal investigated the simplest fractional variational problem and the Lagrange fractional variational problem[6-9]. Baleanu[10-13] did some researches on fractional variational problems for Lagrangian system and Hamiltonin system. The fractional Noether theory was studied by Frederico and Torres[14-16]. Considering that the integration′s lower bound of the functional is different from the fractional derivative′s lower bound of the Lagrangian, Atanacković[17-19] studied a new fractional Lagrange variational problem. The fractional action-like variational problem was presented by El-Nabulsi[20-24]. Though a series of work had been done, they did not include the Birkhoffian system but just the

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* **Corresponding author**: Zhang Yi, Professor, E-mail: weidiezh@gmail.com.

Lagrangian system and the Hamiltonian system.

The fractional Birkhoffian system therefore is studied in this paper. The fractional Pfaff-Birkhoff variational problem is presented in terms of Riesz derivatives. Using the formula for fractional integration by parts and the commutative relations between differential operation and variational operation, the fractional Pfaff-Birkhoff principle and the fractional Birkhoff′s equations are obtained in terms of Riesz-Riemann-Liouville fractional derivatives and Riesz-Caputo fractional derivatives respectively.

2 Preliminaries

In this section some basic necessary facts are presented on the fractional calculus. More details on the subject and its applications can be found in Refs. [2-3].

The left Riemann-Liouville fractional derivative is defined as

$$ {}_{t_1}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_{t_1}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau \tag{1} $$

and the right Riemann-Liouville fractional derivative is

$$ {}_{t}D_{t_2}^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_{t}^{t_2} \frac{f(\tau)}{(\tau-t)^{\alpha-m+1}}\mathrm{d}\tau \tag{2} $$

where $\Gamma(*)$ is the Gamma function and α the order of the fractional derivative, which satisfies $m-1\leqslant\alpha<m$.

The left Caputo fractional derivative is defined as

$$ {}_{t_1}^{\mathrm{C}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_1}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau \quad m-1\leqslant\alpha<m \tag{3} $$

and the right Caputo fractional derivative is

$$ {}_{t}^{\mathrm{C}}D_{t_2}^{\alpha} f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)}\int_{t}^{t_2} \frac{f^{(m)}(\tau)}{(\tau-t)^{\alpha-m+1}}\mathrm{d}\tau \quad m-1\leqslant\alpha<m \tag{4} $$

If α is an integer, these derivatives are defined in the usual sense, i. e.

$$ {}_{t_1}D_t^{\alpha} f(t) = {}_{t_1}^{\mathrm{C}}D_t^{\alpha} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} f(t) \tag{5} $$

$$ {}_{t}D_{t_2}^{\alpha} f(t) = {}_{t}^{\mathrm{C}}D_{t_2}^{\alpha} f(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} f(t) \tag{6} $$

The Riesz-Riemann-Liouville fractional derivative is defined as

$$ {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} f(t) = \frac{1}{2\Gamma(m-\alpha)}\cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_{t_1}^{t_2} \frac{f(\tau)}{|t-\tau|^{\alpha-m+1}}\mathrm{d}\tau \quad m-1\leqslant\alpha<m \tag{7} $$

and the Riesz-Caputo fractional derivative can be written as

$$ {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha} f(t) = \frac{1}{2\Gamma(m-\alpha)}\cdot \int_{t_1}^{t_2} \frac{f^{(m)}(\tau)}{|t-\tau|^{\alpha-m+1}}\mathrm{d}\tau \quad m-1\leqslant\alpha<m \tag{8} $$

If α is an integer, then

$$ {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} f(t) = {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} f(t) \tag{9} $$

The relationship between Riesz-Riemann-Liouville fractional derivative and Riemann-Liouville fractional derivative is

$$ {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} f(t) = \frac{1}{2}\left[{}_{t_1}D_t^{\alpha} f(t) + (-1)^m {}_{t}D_{t_2}^{\alpha} f(t)\right] \tag{10} $$

and the relationship between Riesz-Caputo fractional derivative and Caputo fractional derivative is

$$ {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha} f(t) = \frac{1}{2}\left[{}_{t_1}^{\mathrm{C}}D_t^{\alpha} f(t) + (-1)^m {}_{t}^{\mathrm{C}}D_{t_2}^{\alpha} f(t)\right] \tag{11} $$

The formulae for fractional integration by parts in terms of Riemann-Liouville derivative are[4-5]

$$ \int_{t_1}^{t_2} g(t)\left({}_{t_1}D_t^{\alpha} f(t)\right)\mathrm{d}t = \int_{t_1}^{t_2} f(t)\left({}_{t}D_{t_2}^{\alpha} g(t)\right)\mathrm{d}t + \sum_{k=0}^{m-1}(-1)^k \frac{\mathrm{d}^{m-k-1} f(t)}{\mathrm{d}t^{m-k-1}}\frac{\mathrm{d}^k g(t)}{\mathrm{d}t^k}\bigg|_{t_1}^{t_2} \tag{12} $$

and

$$ \int_{t_1}^{t_2} f(t)\left({}_{t}D_{t_2}^{\alpha} g(t)\right)\mathrm{d}t = \int_{t_1}^{t_2} g(t)\left({}_{t_1}D_t^{\alpha} f(t)\right)\mathrm{d}t - \sum_{k=0}^{m-1}(-1)^k \frac{\mathrm{d}^{m-k-1} f(t)}{\mathrm{d}t^{m-k-1}}\frac{\mathrm{d}^k g(t)}{\mathrm{d}t^k}\bigg|_{t_1}^{t_2} \tag{13} $$

The formulae for fractional integration by parts in terms of Riesz derivatives can be deduced as follows.

Using the relationship between Riesz-Riemann-Liouville fractional derivative and Riemann-Liouville fractional derivative, i. e., Eq. (10), it has

$$\int_{t_1}^{t_2} g(t)\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} f(t)\,\mathrm{d}t = \frac{1}{2}\int_{t_1}^{t_2} g(t)\left[{}_{t_1}D_t^{\alpha} f(t) + (-1)^m {}_tD_{t_2}^{\alpha} f(t)\right]\mathrm{d}t = \frac{1}{2}\int_{t_1}^{t_2} g(t)\, {}_{t_1}D_t^{\alpha} f(t)\,\mathrm{d}t + \frac{1}{2}(-1)^m \cdot \int_{t_1}^{t_2} g(t)\, {}_tD_{t_2}^{\alpha} f(t)\,\mathrm{d}t \tag{14}$$

Substituting Eqs. (12,13) into Eq. (14) obtains

$$\int_{t_1}^{t_2} g(t)\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} f(t)\,\mathrm{d}t = (-1)^m \int_{t_1}^{t_2} f(t)\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} g(t)\,\mathrm{d}t + \frac{1}{2}\sum_{k=0}^{m-1}(-1)^k \frac{\mathrm{d}^{m-k-1} f(t)}{\mathrm{d}t^{m-k-1}}\frac{\mathrm{d}^k g(t)}{\mathrm{d}t^k}\bigg|_{t_1}^{t_2} - \frac{1}{2}\sum_{k=0}^{m-1}(-1)^{k+m} \frac{\mathrm{d}^{m-k-1} g(t)}{\mathrm{d}t^{m-k-1}}\frac{\mathrm{d}^k f(t)}{\mathrm{d}t^k}\bigg|_{t_1}^{t_2} \tag{15}$$

Eq. (15) is the formula for fractional integration by parts in terms of Riesz-Riemann-Liouville derivative.

The formula for fractional integration by parts in terms of Riesz-Caputo derivative is[7]

$$\int_{t_1}^{t_2} g(t)\, {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha} f(t)\,\mathrm{d}t = (-1)^m \int_{t_1}^{t_2} f(t)\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} g(t)\,\mathrm{d}t + \sum_{k=1}^{m-1}(-1)^k\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha+k-m} g(t)\frac{\mathrm{d}^{m-1-k}}{\mathrm{d}t^{m-1-k}} f(t)\bigg|_{t_1}^{t_2} \tag{16}$$

The commutative relations between differential operation and variational operation in terms of Riemann-Liouville fractional derivative are[25]

$$\delta\, {}_{t_1}D_t^{\alpha} F(t,x) = {}_{t_1}D_t^{\alpha}\delta F(t,x) \tag{17}$$

$$\delta\, {}_tD_{t_2}^{\beta} F(t,x) = {}_tD_{t_2}^{\beta}\delta F(t,x) \tag{18}$$

where $F(t,x)$ is an arbitrary function, $x=x(t)$ is the variable which is the function of time t. And similarly we can easily prove the commutative relations between differential operation and variational operation in terms of Caputo fractional derivative, i. e.

$$\delta\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha} F(t,x) = {}_{t_1}^{\mathrm{C}}D_t^{\alpha}\delta F(t,x) \tag{19}$$

$$\delta\, {}_t^{\mathrm{C}}D_{t_2}^{\beta} F(t,x) = {}_t^{\mathrm{C}}D_{t_2}^{\beta}\delta F(t,x) \tag{20}$$

Since the Riesz fractional derivatives are the linear combination of Riemann-Liouvilles or Caputo fractional derivatives, the following relationships are obtained

$$\delta\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} F(t,x) = {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}\delta F(t,x) \tag{21}$$

$$\delta\, {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha} F(t,x) = {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}\delta F(t,x) \tag{22}$$

3 Fractional Pfaff-Birkhoff Principle and Birkhoff′s Equations in Terms of Riesz-Riemann-Liouville Derivative

Consider a Birkhoffian system[26] which is described by $2n$ Birkhoff′s variables $a^\nu (\nu=1,\cdots,2n)$. Suppose the Birkhoffian of the system is $B=B(t,\boldsymbol{a})$, and Birkhoff′s functions are $R_\nu = R_\nu(t,\boldsymbol{a})$. The integral

$$A = \int_{t_1}^{t_2}\left\{\sum_{\nu=1}^{2n} R_\nu(t,\boldsymbol{a})\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} a^\nu - B(t,\boldsymbol{a})\right\}\mathrm{d}t \tag{23}$$

is called the fractional Pfaff-Birkhoff action in terms of Riesz-Riemann-Liouville derivative. Then the isochronous variational principle

$$\delta A = 0 \tag{24}$$

with the commutative relations

$$\delta\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} a^\nu = {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}\delta a^\nu \quad \nu=1,\cdots,2n \tag{25}$$

and the fixed endpoint conditions

$$\delta a^\nu\big|_{t=t_1} = \delta a^\nu\big|_{t=t_2} = 0 \quad \nu=1,\cdots,2n \tag{26}$$

can be called the fractional Pfaff-Birkhoff principle in terms of Riesz-Riemann-Liouville derivatives.

The fractional Birkhoff′s equations can be deduced from the fractional Pfaff-Birkhoff principle. Expanding the principle Eq. (24), it has

$$\delta A = \delta\int_{t_1}^{t_2}\left\{\sum_{\nu=1}^{2n} R_\nu(t,\boldsymbol{a})\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} a^\nu - B(t,\boldsymbol{a})\right\}\mathrm{d}t = \int_{t_1}^{t_2}\delta\left\{\sum_{\nu=1}^{2n} R_\nu(t,\boldsymbol{a})\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} a^\nu - B(t,\boldsymbol{a})\right\}\mathrm{d}t = \int_{t_1}^{t_2}\left\{\sum_{\nu=1}^{2n}\left(\delta R_\nu\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} a^\nu + R_\nu\delta\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} a^\nu\right) - \delta B\right\}\mathrm{d}t = \int_{t_1}^{t_2}\left\{\sum_{\nu=1}^{2n}\left(\sum_{\mu=1}^{2n}\frac{\partial R_\nu}{\partial a^\mu}\delta a^\mu\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} a^\nu + R_\nu\delta\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} a^\nu\right) - \sum_{\mu=1}^{2n}\frac{\partial B}{\partial a^\mu}\delta a^\mu\right\}\mathrm{d}t = 0 \tag{27}$$

that is

$$\delta A = \int_{t_1}^{t_2}\left\{\sum_{\mu=1}^{2n}\left(\sum_{\nu=1}^{2n}\left(\frac{\partial R_\nu}{\partial a^\mu}\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} a^\nu\right) - \frac{\partial B}{\partial a^\mu}\right)\delta a^\mu + \sum_{\nu=1}^{2n}\left(R_\nu\delta\, {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha} a^\nu\right)\right\}\mathrm{d}t = 0 \tag{28}$$

Using the commutative relations Eqs. (25)

and the formula for integration by parts Eq. (15), the equation would be

$$\int_{t_1}^{t_2} R_\nu \delta {}_{t_1}^{R}D_{t_2}^{\alpha} a^\nu \mathrm{d}t = \int_{t_1}^{t_2} R_\nu \, {}_{t_1}^{R}D_{t_2}^{\alpha} \delta a^\nu \mathrm{d}t =$$

$$(-1)^m \int_{t_1}^{t_2} \delta a^\nu \, {}_{t_1}^{R}D_{t_2}^{\alpha} R_\nu \mathrm{d}t +$$

$$\frac{1}{2}\sum_{k=0}^{m-1}(-1)^k \frac{\mathrm{d}^{m-k-1}\delta a^\nu}{\mathrm{d}t^{m-k-1}} \frac{\mathrm{d}^k R_\nu}{\mathrm{d}t^k}\Big|_{t_1}^{t_2} -$$

$$\frac{1}{2}\sum_{k=0}^{m-1}(-1)^{k+m} \frac{\mathrm{d}^{m-k-1} R_\nu}{\mathrm{d}t^{m-k-1}} \frac{\mathrm{d}^k \delta a^\nu}{\mathrm{d}t^k}\Big|_{t_1}^{t_2} \quad (29)$$

Substituting Eq. (29) into Eq. (28), we have

$$\delta A = \int_{t_1}^{t_2} \Big\{ \sum_{\mu=1}^{2n} \Big(\sum_{\nu=1}^{2n} \Big(\frac{\partial R_\nu}{\partial a^\mu} \, {}_{t_1}^{R}D_{t_2}^{\alpha} a^\nu \Big) -$$

$$\frac{\partial B}{\partial a^\mu} + (-1)^m \, {}_{t_1}^{R}D_{t_2}^{\alpha} R_\mu \Big) \delta a^\mu \mathrm{d}t +$$

$$\frac{1}{2}\sum_{\nu=1}^{2n}\sum_{k=0}^{m-1}(-1)^k \frac{\mathrm{d}^{m-k-1}\delta a^\nu}{\mathrm{d}t^{m-k-1}} \frac{\mathrm{d}^k R_\nu}{\mathrm{d}t^k}\Big|_{t_1}^{t_2} -$$

$$\frac{1}{2}\sum_{\nu=1}^{2n}\sum_{k=0}^{m-1}(-1)^{k+m} \frac{\mathrm{d}^{m-k-1} R_\nu}{\mathrm{d}t^{m-k-1}} \frac{\mathrm{d}^k \delta a^\nu}{\mathrm{d}t^k}\Big|_{t_1}^{t_2} = 0 \quad (30)$$

Let

$$\frac{1}{2}\sum_{\nu=1}^{2n}\sum_{k=0}^{m-1}(-1)^k \frac{\mathrm{d}^{m-k-1}\delta a^\nu}{\mathrm{d}t^{m-k-1}} \frac{\mathrm{d}^k R_\nu}{\mathrm{d}t^k}\Big|_{t_1}^{t_2} -$$

$$\frac{1}{2}\sum_{\nu=1}^{2n}\sum_{k=0}^{m-1}(-1)^{k+m} \frac{\mathrm{d}^{m-k-1} R_\nu}{\mathrm{d}t^{m-k-1}} \frac{\mathrm{d}^k \delta a^\nu}{\mathrm{d}t^k}\Big|_{t_1}^{t_2} = 0 \quad (31)$$

Eq. (31) is called the transversality condition in terms of Riesz-Riemann-Liouville fractional derivative. Under the condition Eq. (31), Eq. (30) becomes

$$\delta A = \int_{t_1}^{t_2} \Big\{ \sum_{\mu=1}^{2n} \Big(\sum_{\nu=1}^{2n} \Big(\frac{\partial R_\nu}{\partial a^\mu} \, {}_{t_1}^{R}D_{t_2}^{\alpha} a^\nu \Big) -$$

$$\frac{\partial B}{\partial a^\mu} + (-1)^m \, {}_{t_1}^{R}D_{t_2}^{\alpha} R_\mu \Big) \delta a^\mu \Big\} \mathrm{d}t = 0 \quad (32)$$

According to the arbitrariness of the integral interval $[t_1, t_2]$, the following equation can be obtained.

$$\sum_{\mu=1}^{2n} \Big(\sum_{\nu=1}^{2n} \Big(\frac{\partial R_\nu}{\partial a^\mu} \, {}_{t_1}^{R}D_{t_2}^{\alpha} a^\nu \Big) -$$

$$\frac{\partial B}{\partial a^\mu} + (-1)^m \, {}_{t_1}^{R}D_{t_2}^{\alpha} R_\mu \Big) \delta a^\mu = 0 \quad (33)$$

The principle Eq. (33) is called the fractional Pfaff-Birkhoff-d′Alembert principle in terms of Riesz-Riemann-Liouville fractional derivative. It is a differential variational principle. Because of the independence of δa^μ, one obtains

$$\sum_{\nu=1}^{2n} \Big(\frac{\partial R_\nu}{\partial a^\mu} \, {}_{t_1}^{R}D_{t_2}^{\alpha} a^\nu \Big) - \frac{\partial B}{\partial a^\mu} + (-1)^m \, {}_{t_1}^{R}D_{t_2}^{\alpha} R_\mu = 0$$

$$\mu = 1, \cdots, 2n \quad (34)$$

Eqs. (34) are the fractional Birkhoff′s equations satisfying the transversality condition Eq. (31) in terms of Riesz-Riemann-Liouville fractional derivative.

Using the relationship between the Riesz-Riemann-Liouville fractional derivative and the Riemann-Liouville fractional derivative, Eqs. (34) can be rewritten as

$$\sum_{\nu=1}^{2n} \Big(\frac{\partial R_\nu}{\partial a^\mu} \frac{1}{2} ({}_{t_1}D_t^\alpha a^\nu + (-1)^m \, {}_tD_{t_2}^\alpha a^\nu) \Big) -$$

$$\frac{\partial B}{\partial a^\mu} + (-1)^m \frac{1}{2} ({}_{t_1}D_t^\alpha R_\mu + (-1)^m \, {}_tD_{t_2}^\alpha R_\mu) = 0$$

$$\mu = 1, \cdots, 2n \quad (35)$$

When $\alpha \to 1$, Eqs. (35) become

$$\sum_{\nu=1}^{2n} \Big(\frac{\partial R_\nu}{\partial a^\mu} \frac{1}{2} (\dot{a}^\nu - (-\dot{a}^\nu)) \Big) - \frac{\partial B}{\partial a^\mu} -$$

$$\frac{1}{2} \Big(\frac{\mathrm{d}}{\mathrm{d}t} R_\mu - \Big(-\frac{\mathrm{d}}{\mathrm{d}t} R_\mu \Big) \Big) = 0 \quad \mu = 1, \cdots, 2n$$

Namely,

$$\sum_{\nu=1}^{2n} \Big(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \Big) \dot{a}^\nu - \Big(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} \Big) = 0$$

$$\mu = 1, \cdots, 2n \quad (36)$$

Eqs. (36) are the traditional Birkhoff′s equations. The transversality condition Eq. (31) gives

$$\frac{1}{2}\sum_{\nu=1}^{2n} \delta a^\nu R_\nu \Big|_{t_1}^{t_2} + \frac{1}{2}\sum_{\nu=1}^{2n} R_\nu \delta a^\nu \Big|_{t_1}^{t_2} = 0 \quad (37)$$

i. e.

$$\sum_{\nu=1}^{2n} \delta a^\nu R_\nu \Big|_{t_1}^{t_2} = 0 \quad (38)$$

Using the fixed endpoint conditions Eq. (26), condition Eq. (38) is satisfied. Hence the traditional Birkhoff′s equations Eqs. (36) are special cases of the fractional Birkhoff′s equations Eqs. (34) in terms of Riesz-Riemann-Liouville derivative.

4 Fractional Pfaff-Birkhoff Principle and Birkhoff′s Equations in Terms of Riesz-Caputo Derivative

The integral

$$A = \int_{t_1}^{t_2} \Big\{ \sum_{\nu=1}^{2n} R_\nu(t, \boldsymbol{a}) \, {}_{t_1}^{RC}D_{t_2}^{\alpha} a^\nu - B(t, \boldsymbol{a}) \Big\} \mathrm{d}t \quad (39)$$

is called the fractional Pfaff-Birkhoff action in terms of Riesz-Caputo derivative. Then the isochronous variational principle

$$\delta A = 0 \tag{40}$$

with the commutative relations

$$\delta {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} = {}_{t_1}^{RC}D_{t_2}^{\alpha} \delta a^{\nu} \quad \nu = 1, \cdots, 2n \tag{41}$$

and the fixed endpoint conditions

$$\delta a^{\nu}\big|_{t=t_1} = \delta a^{\nu}\big|_{t=t_2} = 0 \quad \nu = 1, \cdots, 2n \tag{42}$$

can be called the fractional Pfaff-Birkhoff principle in terms of Riesz-Caputo derivatives.

Expanding the principle Eq. (40), it obtains

$$\delta A = \delta \int_{t_1}^{t_2} \Big\{ \sum_{\nu=1}^{2n} R_{\nu}(t, \boldsymbol{a})\, {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} - B(t, \boldsymbol{a}) \Big\} \mathrm{d}t =$$

$$\int_{t_1}^{t_2} \delta \Big\{ \sum_{\nu=1}^{2n} R_{\nu}(t, \boldsymbol{a})\, {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} - B(t, \boldsymbol{a}) \Big\} \mathrm{d}t =$$

$$\int_{t_1}^{t_2} \Big\{ \sum_{\nu=1}^{2n} (\delta R_{\nu}\, {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} + R_{\nu} \delta {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu}) - \delta B \Big\} \mathrm{d}t =$$

$$\int_{t_1}^{t_2} \Big\{ \sum_{\nu=1}^{2n} \Big(\sum_{\mu=1}^{2n} \frac{\partial R_{\nu}}{\partial a^{\mu}} \delta a^{\mu}\, {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} + R_{\nu} \delta {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} \Big) -$$

$$\sum_{\mu=1}^{2n} \frac{\partial B}{\partial a^{\mu}} \delta a^{\mu} \Big\} \mathrm{d}t = 0 \tag{43}$$

i. e

$$\delta A = \int_{t_1}^{t_2} \Big\{ \sum_{\mu=1}^{2n} \Big(\sum_{\nu=1}^{2n} \Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}\, {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} \Big) - \frac{\partial B}{\partial a^{\mu}} \Big) \delta a^{\mu} +$$

$$\sum_{\nu=1}^{2n} R_{\nu} \delta {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} \Big\} \mathrm{d}t = 0 \tag{44}$$

Using the commutative relations Eqs. (41) and the formula for integration by parts Eq. (16), one gets

$$\int_{t_1}^{t_2} R_{\nu} \delta\, {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} \mathrm{d}t = \int_{t_1}^{t_2} R_{\nu}\, {}_{t_1}^{RC}D_{t_2}^{\alpha} \delta a^{\nu} \mathrm{d}t =$$

$$(-1)^{m} \int_{t_1}^{t_2} \delta a^{\nu}\, {}_{t_1}^{R}D_{t_2}^{\alpha} R_{\nu} \mathrm{d}t +$$

$$\sum_{k=0}^{m-1} (-1)^{k}\, {}_{t_1}^{R}D_{t_2}^{\alpha+k-m} R_{\nu} \frac{\mathrm{d}^{m-1-k} \delta a^{\nu}}{\mathrm{d}t^{m-1-k}} \Bigg|_{t_1}^{t_2} \tag{45}$$

Substituting Eq. (45) into Eq. (44), we have

$$\delta A = \int_{t_1}^{t_2} \Big\{ \sum_{\mu=1}^{2n} \Big(\sum_{\nu=1}^{2n} \Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}\, {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} \Big) - \frac{\partial B}{\partial a^{\mu}} +$$

$$(-1)^{m}\, {}_{t_1}^{R}D_{t_2}^{\alpha} R_{\mu} \Big) \delta a^{\mu} \Big\} \mathrm{d}t + \sum_{\nu=1}^{2n} \sum_{k=0}^{m-1} (-1)^{k} \cdot$$

$${}_{t_1}^{R}D_{t_2}^{\alpha+k-m} R_{\nu} \frac{\mathrm{d}^{m-1-k} \delta a^{\nu}}{\mathrm{d}t^{m-1-k}} \Bigg|_{t_1}^{t_2} = 0 \tag{46}$$

Let

$$\sum_{\nu=1}^{2n} \sum_{k=0}^{m-1} (-1)^{k}\, {}_{t_1}^{R}D_{t_2}^{\alpha+k-m} R_{\nu} \frac{\mathrm{d}^{m-1-k} \delta a^{\nu}}{\mathrm{d}t^{m-1-k}} \Bigg|_{t_1}^{t_2} = 0 \tag{47}$$

Eq. (47) is the transversality condition in terms of Riesz-Caputo fractional derivative. From the condition Eq. (47), Eq. (46) becomes

$$\delta A = \int_{t_1}^{t_2} \Big\{ \sum_{\mu=1}^{2n} \Big(\sum_{\nu=1}^{2n} \Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}\, {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} \Big) - \frac{\partial B}{\partial a^{\mu}} +$$

$$(-1)^{m}\, {}_{t_1}^{R}D_{t_2}^{\alpha} R_{\mu} \Big) \delta a^{\mu} \Big\} \mathrm{d}t = 0 \tag{48}$$

According to the arbitrariness of the integral interval $[t_1, t_2]$, one obtains

$$\sum_{\mu=1}^{2n} \Big(\sum_{\nu=1}^{2n} \Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}\, {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} \Big) - \frac{\partial B}{\partial a^{\mu}} +$$

$$(-1)^{m}\, {}_{t_1}^{R}D_{t_2}^{\alpha} R_{\mu} \Big) \delta a^{\mu} = 0 \tag{49}$$

The principle Eq. (49) is called the fractional Pfaff-Birkhoff-d′ Alembert principle in terms of Riesz-Caputo fractional derivative. Because of the independence of δa^{μ}, it obtains

$$\sum_{\nu=1}^{2n} \Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}\, {}_{t_1}^{RC}D_{t_2}^{\alpha} a^{\nu} \Big) - \frac{\partial B}{\partial a^{\mu}} + (-1)^{m}\, {}_{t_1}^{R}D_{t_2}^{\alpha} R_{\mu} = 0$$

$$\mu = 1, \cdots, 2n \tag{50}$$

Eqs. (50) are the fractional Birkhoff′s equations satisfying the transversality condition Eq. (47) in terms of Riesz-Caputo fractional derivative.

Using the relationships between Riesz fractional derivative, Riemann-Liouville fractional derivative and Caputo fractional derivative, Eqs. (50) become

$$\sum_{\nu=1}^{2n} \Big(\frac{\partial R_{\nu}}{\partial a^{\mu}} \frac{1}{2} ({}_{t_1}^{C}D_{t}^{\alpha} a^{\nu} + (-1)^{m}\, {}_{t}^{C}D_{t_2}^{\alpha} a^{\nu}) \Big) -$$

$$\frac{\partial B}{\partial a^{\mu}} + (-1)^{m} \frac{1}{2} ({}_{t_1}D_{t}^{\alpha} R_{\mu} + (-1)^{m}\, {}_{t}D_{t_2}^{\alpha} R_{\mu}) = 0$$

$$\mu = 1, \cdots, 2n \tag{51}$$

When $\alpha \to 1$, Eqs. (51) can be written as

$$\sum_{\nu=1}^{2n} \Big(\frac{\partial R_{\nu}}{\partial a^{\mu}} \frac{1}{2} (\dot{a}^{\nu} - (-\dot{a}^{\nu})) \Big) - \frac{\partial B}{\partial a^{\mu}} -$$

$$\frac{1}{2} \Big(\frac{\mathrm{d}}{\mathrm{d}t} R_{\mu} - \Big(-\frac{\mathrm{d}}{\mathrm{d}t} R_{\mu} \Big) \Big) = 0 \quad \mu = 1, \cdots, 2n$$

Namely,

$$\sum_{\nu=1}^{2n} \Big(\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}} \Big) \dot{a}^{\nu} - \Big(\frac{\partial B}{\partial a^{\mu}} + \frac{\partial R_{\mu}}{\partial t} \Big) = 0$$

$$\mu = 1, \cdots, 2n \tag{52}$$

Eqs. (52) are the traditional Birkhoff′s equations. The transversality condition Eq. (47) gives

$$\sum_{\nu=1}^{2n} {}_{t_1}^{R}D_{t_2}^{\alpha-1} R_{\nu} \delta a^{\nu} \Bigg|_{t_1}^{t_2} = 0 \tag{53}$$

Using the fixed endpoint conditions Eqs. (42), condition Eq. (53) can be satisfied. Hence the traditional Birkhoff′ s equations

Eqs. (52) are special cases of the fractional Birkhoff's equations Eqs. (50) in terms of Riesz-Caputo derivative.

5 Illustrative Example

In order to illustrate the above results, a Birkhoffian system is studied whose Birkhoffian and Birkhoff's functions are

$$B=\frac{1}{2}[(a^1)^2+(a^3)^2+(a^4)^2] \quad (54)$$

$$R_1=a^3,\ R_2=a^4,\ R_3=R_4=0 \quad (55)$$

Authors try to establish the fractional Birkhoff's equations in terms of Riesz-Riemann-Liouville fractional derivative and Riesz-Caputo fractional derivative.

Without loss of generality, it is supposed that $0<\alpha<1$. The fractional Pfaff-Birkhoff action in terms of Riesz-Riemann-Liouville derivative Eq. (23) gives

$$A=\int_{t_1}^{t_2}\left\{a^3\,{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^1+a^4\,{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^2-\frac{1}{2}[(a^1)^2+(a^3)^2+(a^4)^2]\right\}\mathrm{d}t \quad (56)$$

The fractional Pfaff-Birkhoff-d'Alembert principle in terms of Riesz-Riemann-Liouville fractional derivative Eq. (33) gives

$$(-a^1-{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^3)\delta a^1+({}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^4)\delta a^2+({}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^1-a^3)\delta a^3+({}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^2-a^4)\delta a^4=0 \quad (57)$$

According to the independence of δa^{ν} ($\nu=1,2,3,4$), the fractional Birkhoff's equations Eqs. (34) corresponding to the action Eq. (56) are obtained as follows

$$-a^1-{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^3=0,\quad {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^4=0$$
$${}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^1-a^3=0,\ {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^2-a^4=0 \quad (58)$$

and the transversality condition Eq. (38) gives

$$(a^3\delta a^1)\big|_{t_1}^{t_2}+(a^4\delta a^2)\big|_{t_1}^{t_2}=0 \quad (59)$$

The fractional Pfaff-Birkhoff action in terms of Riesz-Caputo derivative Eq. (39) gives

$$A=\int_{t_1}^{t_2}\left\{a^3\,{}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}a^1+a^4\,{}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}a^2-\frac{1}{2}[(a^1)^2+(a^3)^2+(a^4)^2]\right\}\mathrm{d}t \quad (60)$$

The fractional Pfaff-Birkhoff-d'Alembert principle in terms of Riesz-Caputo fractional derivative Eq. (49) gives

$$(-a^1-{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^3)\delta a^1+({}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^4)\delta a^2+({}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}a^1-a^3)\delta a^3+({}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}a^2-a^4)\delta a^4=0 \quad (61)$$

Eqs. (50) yield

$$-a^1-{}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^3=0,\ {}^{\mathrm{R}}_{t_1}D^{\alpha}_{t_2}a^4=0$$
$${}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}a^1-a^3=0,\ {}^{\mathrm{RC}}_{t_1}D^{\alpha}_{t_2}a^2-a^4=0 \quad (62)$$

Eqs. (62) are the fractional Birkhoff's equations in terms of Riesz-Caputo derivative, and the transversality condition Eq. (53) gives

$$({}^{\mathrm{R}}_{t_1}D^{\alpha-1}_{t_2}a^3\delta a^1)\Big|_{t_1}^{t_2}-({}^{\mathrm{R}}_{t_1}D^{\alpha-1}_{t_2}a^4\delta a^2)\Big|_{t_1}^{t_2}=0 \quad (63)$$

6 Conclusions

In 1996, Riewe applied the fractional calculus to dynamics modeling of non-conservative mechanical systems, and presented the issue of fractional variational problem for the first time. Now the fractional variational problems are attractive topics in the fields of mathematics, mechanics and physics. A number of important results have been achieved. In this paper, a further study is conducted on the fractional Pfaff-Birkhoff variational problems based on the Riesz fractional derivatives. The fractional Pfaff-Birkhoff principle is presented, and the fractional Pfaff-Birkhoff-d'Alembert principle and the fractional Birkhoff's equations in terms of Riesz fractional derivatives are established. The results in this paper are of universal significance. Besides, the traditional Pfaff-Birkhoff principle and Birkhoff's equations under the integer order derivatives are the special cases of this paper.

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