MAC Models in Thermoelasticity

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Abstract: There are two types of singularities in the linear thermoelasticity. The first one arises in the field of stresses if a force is applied to one point of the body. This singularity is physical and should be accepted. The second type of singularities is nonphysical and they arise in the fields of displacements and temperatures. There exist the nonlocal theories and gradient theories which have the goal to introduce the finite stresses instead of the infinite ones. The MAC model of the thermoelasticity is created to avoid the nonphysical singularities and it accepts the infinite stresses. MAC is the method of additional conditions, which allows introducing the new model to use the classical model, plus additional condition of the physical nonsingularity and/or condition of the good behavior of the solutions at infinity. The MAC Green's functions for the heat conduction and for the elasticity could be introduced using the differential MAC models. The infinite and finite bodies are considered. The principle of superposition is applied to obtain the integral equations to solve the boundary value problems. The strength criteria based on finite stresses could be changed in this model because the infinite stresses are allowed. The strength criteria based on deformations are applicable. Classification of MAC models is given.

Key words: MAC model; constitutive equations; thermoelasticity

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1 Introduction

The classical continuum mechanics considers constitutive equations for the stresses. However the equations of motion of the small control volume of the continuum media include not only stress-vectors applied to the surface of an element but also the body-forces as in Refs. [1,2]. Then it is not logical to avoid the constitutive equations for the body-forces.

Different examples of the constitutive equations for the body-forces are considered in this paper. The obtained mathematical models are called the MAC models. MAC is called the method of additional conditions[3], and the additional condition in this paper is the second constitutive equation for the body-forces.

The two constitutive equations could be considered as a point in the plane with two coordinate axes. The first one is the axis for stresses and the points on this axis different from existing theories like elasticity, plasticity etc. The second one is the axis for the body-forces and the points on that axis different from constitutive equations for the body-forces such as elasticity, plasticity, and many others. Then the power to analyze theoretically the behavior of the continuum body will be now

$$N^2 \tag{1}$$

where N is the number of existing constitutive equations for stresses. Moreover the third constitutive equation for the body moments could be introduced, and then the theoretical power to analyze the continuum will be of the order N^3. But the third constitutive equation is not applied in this paper.

For example, the constitutive equation for an elastic body could be taken for stresses, and the constitutive equations for viscous flow are used for the body-forces.

The simplest theories of an elastic body and of the heat conduction are presented in this paper

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to show some properties of the models with two constitutive equations.

The MAC models for thermoelasticity with one constitutive equation are considered in Ref. [4].

2 Tension of an Elastic Bar

2.1 Statement of the problem

Consider the simple tension of an elastic bar, where the initial or the reference frame of a bar corresponds to the zero displacements, strains and stresses in the bar.

That state of a bar could be found using the known temperature distribution and the Duhamel-Neumann law [2]. This law gives the distribution of strains in the body, when the stresses are zero and the temperature distribution is available. Then the displacements will be found using the known strains. That is the natural initial state of the body, which is taken above.

The equation of one-dimensional motion of a bar is

$$\frac{\partial N}{\partial x}+q_0+q_1=\rho\frac{\partial^2 u}{\partial t^2} \tag{2}$$

where N is the normal force applied to the cross-section of a bar, x a Cartesian coordinate of a cross-section, and $0<x<L$, L the length of the bar, u the longitudinal displacement of the transversal cross-section of the bar, t the time, q_0 an elastic body forces reaction, and q_1 the density of the external longitudinal body forces per unit length. It is assumed that the body force q is the sum of two parts, where the first part depends on the state of deformation of the body, and the second part of the body forces represents the external given forces like gravitational forces, that is

$$q=q_0+q_1 \tag{3}$$

The Hook law is

$$N=EA\varepsilon \tag{4}$$

where E is the Young modulus, A the constant cross-sectional area, and ε the longitudinal strain, which is supposed to be small

$$\varepsilon=\frac{\partial u}{\partial x} \tag{5}$$

Substituting Eqs. (4,5) into Eq. (2), the following equation will be obtained

$$c^2\frac{\partial^2 u}{\partial x^2}+p_0+p_1=\frac{\partial^2 u}{\partial t^2} \tag{6}$$

where

$$c^2=\frac{EA}{\rho},\ p_0=\frac{q_0}{\rho},\ p_1=\frac{q_1}{\rho} \tag{7}$$

2.2 Classical solution

Consider the steady state problem for a bar as a particular problem. Let the external distributed force p_1 be not given and the inertial term be neglected. If the second constitutive equation is not taken, $p_0=0$, and Eq. (6) becomes

$$\frac{\partial^2 u}{\partial x^2}=0 \tag{8}$$

Consider the boundary conditions

$$u(0)=u_0\neq 0,\ u(L)=0 \tag{9}$$

The general solution of Eq. (8) is

$$u(x)=Ax+B \tag{10}$$

where A, B are arbitrary constants. If the length L of the bar is limited, the solution of the problem Eqs. (8,9) is

$$u=u_0\left(1-\frac{x}{L}\right) \tag{11}$$

If the length of the bar is infinite, the two discontinuous solutions could be obtained. The first one follows from the solution Eq. (11) for the finite bar as a limit $L\to\infty$. That is

$$u(x)=u_0\quad 0\leqslant x<\infty \tag{12}$$

$$u(\infty)=0 \tag{13}$$

The second solution for an infinite bar will be obtained if we take the general solution Eq. (10) and satisfy the second condition of the Eq. (9) at infinity. Then we get

$$A=0,\ B=0 \tag{14}$$

and the second solution becomes

$$u(0)=u_0 \tag{15}$$

$$u(x)=0\qquad 0<x\leqslant\infty \tag{16}$$

The situation for unlimited bar is undetermined because there are two absolutely different and discontinuous solutions Eqs. (12,13) and Eqs. (15,16). It seems that the limit solution (Eqs. (12,13)) for the finite bar is the most desirable. But there exist the theories, for which the solution (Eqs. (15, 16)) is a desirable limit solution. The equations below are taken form Ref. [5].

2.2.1 Mindlin′s theory

The one dimensional equation corresponding to the bar is

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}-[l_1^2(\lambda+\mu)+l_2^2\mu]\frac{\partial^4 u}{\partial x^4}=0 \tag{17}$$

where λ and μ are the Lame constants, l_1 and l_2 the internal length scale parameters that account for the microstructural effects.

The solution for the infinite bar will be

$$u=u_0\mathrm{e}^{-kx} \tag{18}$$

where

$$k=\sqrt{\frac{\lambda+2\mu}{l_1^2(\lambda+\mu)+l_2^2\mu}} \tag{19}$$

The constants l_1 and l_2 are small parameters according to their physical sense. Then the solution Eq. (18) corresponds to Eqs. (15,16).

2.2.2 Eringen′s theory

This theory creates the following equation

$$\frac{\partial^2 u}{\partial x^2}=0 \tag{20}$$

This equation coincides with classical one, that is considered before in the Section 2.2.

2.2.3 Aifantis theory

This theory creates the equation

$$\left(1-l^2\frac{\partial^2}{\partial x^2}\right)C_{1111}\frac{\partial^2 u}{\partial x^2}=0 \tag{21}$$

where l replaces l_1 and l_2 in Mindlin′s theory, C_{1111} is a component of the constitutive tensor. This theory could be considered as a simplification of the Mindlin model. Then the required solution for the infinite bar is

$$u=u_0\mathrm{e}^{-\frac{x}{l}} \tag{22}$$

The solution Eq. (22) is an approximation to the solution Eqs. (15, 16), because the parameter l is small.

2.3 MAC model

Let us consider one of the MAC models of the tension of an elastic bar according to Ref. [2]. The linear term is introduced into Eq. (8)

$$\frac{\partial^2 u}{\partial x^2}-au=0 \quad 0<x<\infty \tag{23}$$

where $a>0$ is a parameter, which should be determined in experiment.

Then the solution for an infinite bar will be

$$u=u_0\mathrm{e}^{-\sqrt{a}x} \tag{24}$$

If the additional term in Eq. (23) is supposed to be small, the solution Eq. (23) is an approximation to the solution Eqs. (12,13).

The model (23) could be considered from two points of view. The first one corresponds to the model with one constitutive equation, and the correspondent Hook law is

$$\frac{\partial N}{\partial x}=EA\frac{\partial \varepsilon}{\partial x}-EAau \tag{25}$$

The second point of view corresponds to the model with two constitutive equations, where the constitutive equation for body forces creates

$$q_0=-au \tag{26}$$

3 Linear isotropic elasticity

3.1 MAC model 1

The equations of motion for the linearly isotropic elastic solid are given according to Ref. [6] in the following form

$$\rho_0\frac{\partial^2 \boldsymbol{u}}{\partial t^2}=\rho_0\boldsymbol{B}+(\lambda+\mu)\nabla e+\mu\nabla^2\boldsymbol{u} \tag{27}$$

where dilatation e is

$$e=\mathrm{div}\boldsymbol{u} \tag{28}$$

and $\boldsymbol{u}$ the displacement vector, which does not include the free thermal displacements, ρ_0 the density, $\rho_0\boldsymbol{B}$ the body force per unit volume, λ and μ are Lame′s coefficients or Lame′s constants, ∇ is the gradient, and ∇^2 the Laplacian. The variables x_1, x_2, x_3 are Cartesian coordinates of a point belonging to the domain Ω.

Let the body forces be the sum of two parts

$$\rho_0\boldsymbol{B}=\rho_0\boldsymbol{C}+\rho_0\boldsymbol{D} \tag{29}$$

where $\rho_0\boldsymbol{C}$ is the external body forces per unit volume, and $\rho_0\boldsymbol{D}$ the internal elastic body forces per unit volume. Let

$$\rho_0\boldsymbol{D}=-d\boldsymbol{u} \tag{30}$$

where d is a constant, which should be determined experimentally. Then Eq. (27) will take the form

$$\rho_0\frac{\partial^2 \boldsymbol{u}}{\partial t^2}=\rho_0\boldsymbol{C}-d\boldsymbol{u}+(\lambda+\mu)\nabla e+\mu\nabla^2\boldsymbol{u} \tag{31}$$

Eq. (31) is the equation of the MAC model 1 for the linearly isotopic elasticity. This equation could be considered as a generalization of Eq. (6) for a rod. An important property of Eq. (31) is the real behavior of the displacements at infinity for 2D elastic problems. The classical equations can show the unbounded growth of displacements at infinity.

3.2 MAC model 2

If a force is applied to one point of an elastic body then Eq. (31) allows the infinite displacements at that point. It seems to be not true for the real body, because the force could be applied to one molecule or to one atom and displacements of that particles are finite. The goal to obtain the finite solutions in the described situation can be reached using the differential MAC models in Ref. [2]. Then the body forces include an additional term as a result of interaction of the applied force and the state of deformation of the body.

The equations of the MAC model 2 for an infinite linearly isotropic elastic body will take the following form

$$(\lambda+\mu)\frac{\partial e}{\partial x_i}+\mu\nabla^2 u_i-du_i=\iiint_\Omega\Big[\frac{\mu(x_{i+1}-y_{i+1})}{(x_{i+1}-y_{i+1})^2+(x_{i+2}-y_{i+3})^2}\frac{\partial u_i}{\partial x_{i+1}}(\boldsymbol{x}-\boldsymbol{y},t)+\frac{\mu(x_{i+2}-y_{i+2})}{(x_{i+1}-y_{i+1})^2+(x_{i+2}-y_{i+2})^2}\frac{\partial u_i}{\partial x_{i+2}}(\boldsymbol{x}-\boldsymbol{y},t)+\rho_0\frac{\partial^2 u_i}{\partial t^2}(\boldsymbol{x}-\boldsymbol{y},t)-\rho_0 C_i(\boldsymbol{x}-\boldsymbol{y},t)\Big]\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3 \tag{32}$$

There exists another way to get the finite displacements under the applied force. It is possible to put the term of higher order into the expression of body forces. Then the gradient theories with two constitutive equations should be considered. But this model is not considered in this paper.

4 Heat Conduction

4.1 MAC model 1

Consider the classical heat conduction problem with the equation according to Ref. [4]

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+q_0+q_1=c_0\rho\frac{\partial u}{\partial t} \tag{33}$$

where x,y,z are Cartesian coordinates, t is time, $u(x,y,z,t)$ the temperature, $\rho(x,y,z)$ the mass-density of the body per unit volume, c_0 the specific heat, k the coefficient of thermal conduction, q_0 a rate of internal heat reaction per unit volume, and q_1 a rate of internal heat generation per unit volume, produced in the body. The introduced in Eq. (33) term q_0 corresponds to the body forces reaction, considered above in the elastic body.

The different MAC models will be obtained for the different constitutive laws describing q_0. The MAC model 1 could be considered for

$$q_0=-\alpha u \tag{34}$$

where α is an additional parameter, which could be determined experimentally. Then Eqs. (33,34) will give

$$k\nabla^2 u-\alpha u+q_1=c_0\rho\frac{\partial u}{\partial t} \tag{35}$$

The MAC model 1 will exclude the nonphysical growth of the temperature for infinite body in 2D heat conduction problems.

4.2 MAC model 2

Let

$$q_0=-\alpha u-\beta\nabla^4 u \tag{36}$$

where α,β are the parameters, which could be determined experimentally. Consider the following equation

$$-\beta\nabla^4 u+k\nabla^2 u-\alpha u+q_1=c_0\rho\frac{\partial u}{\partial t} \tag{37}$$

Eq. (37) does not have a singularity in temperature in 2D case of a source at some point, and the strength of the source is finite. Consider a steady state problem for an infinite body with the axis of symmetry and without internal sources. Eq. (37) in cylindrical coordinates will be

$$-\beta\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right)\right)^2+\frac{k}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right)-\alpha u=0 \tag{38}$$

The boundary conditions are taken in the form

$$u(0)=u_0\neq 0,\ u(\infty)=0 \tag{39}$$

The general solution of the Eq. (38) is

$$u(r)=C_1 I_0(\lambda_1 r)+C_2 K_0(\lambda_1 r)+C_3 I_0(\lambda_2 r)+C_4 K_0(\lambda_2 r) \tag{40}$$

where C_1,C_2,C_3,C_4 are arbitrary constants, the parameters λ_1 and λ_2 equal

$$\lambda_{1,2}=\frac{k\pm\sqrt{k^2-4\alpha\beta}}{2\beta}>0 \tag{41}$$

If $C_1 = C_3 = 0$, then the second condition of the Eq. (39) will be fulfilled. The first condition in Eq. (39) is satisfied in case

$$C_2 = -C_4 = \frac{2u_0}{\ln \frac{\lambda_2}{\lambda_1}} \tag{42}$$

Then the solution of the stated problem (Eqs. (38,39)) is

$$u(r) = \frac{2u_0}{\ln \frac{\lambda_2}{\lambda_1}} \left[K_0(\sqrt{\lambda_1}\, r) - K_0(\sqrt{\lambda_2}\, r) \right] \tag{43}$$

The MAC model 2 gives the finite temperatures of the source, and the strength of the source is also finite.

The MAC models of the type Eq. (37) are of the higher order differential equations, but they seem to be more simple as the MAC models of the type Eq. (32), which of that models is the best with respect to reality could not be decided in this paper.

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