General Solutions of Thermoelastic Plane Problems of Two-Dimensional Quasicrystals

Zhang Liangliang(张亮亮)[1,2], *Yang Lianzhi*(杨连枝)[1,2], *Yu Lianying*(余莲英)[1], *Gao Yang*(高阳)[1*]

1. College of Science, China Agricultural University, 100083, Beijing, P. R. China;
2. College of Engineering, China Agricultural University, 100083, Beijing, P. R. China

(Received 4 November 2013; revised 24 December 2013; accepted 25 January 2014)

Abstract: The thermoelastic plane problems of two-dimensional decagonal quasicrystals (QCs) are systematically investigated. By introducing a displacement function, the problem of thermoelastic plane problems can be simplified to an eighth-order partial differential governing equation, and then general solutions are presented through an operator method. By virtue of the Almansi′s theorem, the general solutions are further established, and all expressions for the phonon, phason and thermal fields are described in terms of the potential functions. As an application of the general solution, for a steady point heat source in a semi-infinite quasicrystal plane, the closed form solutions are presented by four newly induced harmonic functions.

Key words: two-dimensional quasicrystals; thermoelasticity; general solutions; point heat source

CLC number: O34; O469 **Document code**: A **Article ID**: 1005-1120(2014)02-0142-05

1 Introduction

An increasing number of quasicrystals (QCs) with good thermal stability make thermoelasticity analyses for QCs more and more important. Furthermore, in view of the fact that QCs have a potential to be used as the components in drilling and nuclear storage facilities[1], it′s very necessary to study the influence of the temperature for QCs. For the general solutions of QCs, thermal effort is always beyond the scope of the studies. Wang et al[2] derived the general solutions for thermoelastic problems of two-dimensional (2D) decagonal QCs by using the complex variable technique. Li et al[3] inferred the general solutions for three-dimensional (3D) thermoelastic problems of one-dimensional (1D) hexagonal QCs. For plane piezothermoelastic medium, Xiong et al[4] and Kumar et al[5] deduced the general steady state solution, respectively. The general solutions for 2D plane thermoelastic problems, such as 2D decagonal QCs have not been attempted. The purpose of this paper is to systematically investigate the thermoelastic plane problems of two-dimensional decagonal QCs.

2 Basic Equations

2D QCs refer to a 3D solid structure with two quasi-periodic arrangement directions and one periodic direction. For 2D QCs with Cartesian coordinate system (x_1, x_2, x_3), we assume that x_1-x_2 is the quasi-periodic plane and x_3 is the periodic direction. Due to the mathematical complexities, 3D problems of 2D QCs are difficult to be analytically solved. In order to explicitly study the phonon-phason interaction, only plane elasticity theory is considered in this paper. Since the x_1-x_2 plane is the quasi-periodic plane, we can employ the x_1-x_3 plane or the x_2-x_3 plane for the study of plane phonon-phason coupling phenomena. In the present work we chose the former, so this problem can be decomposed into an anti-plane problem and an in-plane problem. In this paper, only in-plane problem is considered, the field var-

* **Corresponding author**: Gao Yang, Professor, E-mail: gaoyangg@gmail.com.

iables are independent of x_2, such that $\partial_2 = 0$, where $\partial_j = \partial/\partial_{x_j}$. For 2D decagonal QCs, the point groups 10mm, 1022, $\bar{1}0$m2, 10 /mmm belong to Laue class 14. In the absence of body forces, the general equations governing the plane 2D decagonal QCs can be written as

$$\varepsilon_{ij} = 0.5(\partial_j u_i + \partial_i u_j), \quad w_{1j} = \partial_j w_1 \tag{1}$$

$$\partial_j \sigma_{ij} = 0, \partial_j H_{1j} = 0 \qquad i,j = 1,3 \tag{2}$$

$$\begin{cases} \sigma_{11} = C_{11}\varepsilon_{11} + C_{13}\varepsilon_{33} + Rw_{11} - \beta_1 T \\ \sigma_{33} = C_{13}\varepsilon_{11} + C_{33}\varepsilon_{33} - \beta_3 T \\ \sigma_{13} = \sigma_{31} = 2C_{44}\varepsilon_{31} \\ H_{11} = R\varepsilon_{11} + K_1 w_{11}, H_{13} = K_4 w_{13} \end{cases} \tag{3}$$

where u_i and w_1 denote phonon and phason displacements in the physical and perpendicular spaces, respectively; σ_{ij} and ε_{ij} are the phonon stresses and strains; H_{1j} and w_{1j} are the phason stresses and strains; $C_{11}, C_{13}, C_{33}, C_{44}$ represent the elastic constants in phonon field; K_1, K_4 are the elastic constants in phason field; R is the phonon-phason coupling elastic constants; β_1, β_3 are the thermal constants; T is the variation of the temperature. By virtue of the parallel method proposed by Gao et al[6], the equilibrium equation can be represented with u_i and w_1 as follows

$$\begin{cases} (C_{11}\partial_1^2 + C_{44}\partial_3^2)u_1 + (C_{13}+C_{44})\partial_1\partial_3 u_3 + R\partial_1^2 w_1 - \\ \qquad \beta_1\partial_1 T = 0 \\ (C_{13}+C_{44})\partial_1\partial_3 u_1 + (C_{44}\partial_1^2 + C_{33}\partial_3^2)u_3 - \beta_3\partial_3 T = 0 \\ R\partial_1^2 u_1 + (K_1\partial_1^2 + K_4\partial_3^2)w_1 = 0 \end{cases} \tag{4}$$

Assuming that the thermoelastic loading changes slowly with time and without consideration of the rate of entropy, the uncoupled thermoelastic theory of QCs is adopted in the following analysis. In a steady-state, the heat conductivity equation is

$$(k_{11}\partial_1^2 + k_{33}\partial_3^2)T = 0 \tag{5}$$

where k_{11} and k_{33} are thermal conductivity coefficients. In a similar manner to transversely isotropic elasticity[7] and piezoelasticity[8], the balance Eq. (4) and heat conductivity Eq. (5) for the problem is

$$\boldsymbol{AU} = \boldsymbol{0} \tag{6}$$

where the vector $\boldsymbol{U} = [u_1, u_3, w_1, T]^{\mathrm{T}}$ (the superscript "T" denotes the transpose); $\boldsymbol{A}$ is a 4×4 differential operator matrix, such as

$$\boldsymbol{A} = \begin{bmatrix} C_{11}\partial_1^2 + C_{44}\partial_3^2 & (C_{13}+C_{44})\partial_1\partial_3 & R\partial_1^2 & -\beta_1\partial_1 \\ (C_{13}+C_{44})\partial_1\partial_3 & C_{44}\partial_1^2 + C_{33}\partial_3^2 & 0 & -\beta_3\partial_3 \\ R\partial_1^2 & 0 & K_1\partial_1^2 + K_4\partial_3^2 & 0 \\ 0 & 0 & 0 & k_{11}\partial_1^2 + k_{33}\partial_3^2 \end{bmatrix}$$

In terms of Eq. (6), it seems to be extremely difficult to find the solution by means of direct integration due to the complexity of the equations. Furthermore, a decomposition and superposition procedure is manipulated to simplify the complicated governing equation by introducing a displacement function.

3 General Solutions of Problem

By virtue of the operator analysis technique[6,9-10], the general solutions of the problem will be developed. Introduce a 4×4 differential operator matrix $\boldsymbol{B}$, components B_{ij} of which are "algebraic complement minors" of $\boldsymbol{A}$ in Eq. (4), i. e.

$$\boldsymbol{AB} = \boldsymbol{BA} = A_0 \boldsymbol{I}$$

where A_0 is the "determinant" of the differential operator $\boldsymbol{A}$, $\boldsymbol{I}$ the unit matrix. Then the general solution of Eq. (5) can be expressed as

$$\boldsymbol{U} = \boldsymbol{B\varphi}$$

where the displacement function vector $\boldsymbol{\varphi}$ satisfies the following equation

$$A_0 \boldsymbol{\varphi} = 0$$

The "determinant" A_0 of $\boldsymbol{A}$ yields

$$A_0 = (k_{11}\partial_1^2 + k_{33}\partial_3^2)(a\partial_3^6 + b\partial_1^2\partial_3^4 + c\partial_1^4\partial_3^2 + d\partial_1^6) \tag{7}$$

where

$a = C_{33}C_{44}K_4$

$b = C_{33}C_{44}K_1 + (C_{11}C_{33} - C_{13}^2 - 2C_{13}C_{44})K_4$

$c = C_{11}C_{44}K_4 + (C_{11}C_{33} - C_{13}^2 - 2C_{13}C_{44})K_1 - C_{33}R^2$

$d = C_{11}C_{44}K_1 - C_{44}R^2$

Now introduce a displacement function H, which satisfies

$$\nabla_1^2\nabla_2^2\nabla_3^2\nabla_4^2 H = 0 \tag{8}$$

where the quasi-harmonic differential operators ∇_q^2 are expressed as

$$\nabla_q^2 = \partial_1^2 + \partial_3^2/s_q^2$$

Here $q = 1, 2, 3, 4$, $s_4^2 = k_{11}/k_{33}$, s_1^2, s_2^2 and s_3^2 are the three characteristic roots (or eigenvalues) of the following cubic algebra equation

$$as^6 - bs^4 + cs^2 - d = 0 \tag{9}$$

If the index q is taken to be 1, 2 or 3, three sets of general solutions with $T=0$ will be obtained, which are actually the elastic general solutions without thermal effect. Taking $q=4$, we can obtain

$$u_1 = B_{41}H, u_3 = B_{42}H, w_1 = B_{43}H, T = B_{44}H \tag{10}$$

or

$$u_1 = (a_1\partial_3^4 + b_1\partial_1^2\partial_3^2 + c_1\partial_1^4)\partial_1 H$$
$$u_3 = (a_2\partial_3^4 + b_2\partial_1^2\partial_3^2 + c_2\partial_1^4)\partial_3 H$$
$$w_1 = (b_3\partial_1^2\partial_3^2 + c_3\partial_1^4)\partial_3 H$$
$$T = (a\partial_3^6 + b\partial_1^2\partial_3^4 + c\partial_1^4\partial_3^2 + d\partial_1^6)H$$

where

$$a_1 = -C_{33}K_4\beta_1 + (C_{13} + C_{44})K_4\beta_3$$
$$b_1 = -(C_{33}K_1 + C_{44}K_4)\beta_1 + (C_{13} + C_{44})K_1\beta_3$$
$$c_1 = -C_{44}K_1\beta_1, a_2 = C_{44}K_4\beta_3, a_3 = 0$$
$$b_2 = -(C_{13} + C_{44})K_4\beta_1 + (C_{44}K_1 + C_{11}K_4)\beta_3$$
$$c_2 = -(C_{13} + C_{44})K_1\beta_1 + (C_{11}K_1 - R^2)\beta_3$$
$$b_3 = C_{33}R\beta_1 - (C_{13} + C_{44})R\beta_3, c_3 = C_{44}R\beta_1$$

By utilizing the generalized Almansi′s theorem[11] the displacement function H can be expressed by four quasi-harmonic equations H_q in five distinct forms as

$$H = \begin{cases} H_1 + H_2 + H_3 + H_4 & s_1^2 \neq s_2^2 \neq s_3^2 \neq s_4^2 \\ H_1 + H_2 + H_3 + x_3 H_4 & s_1^2 \neq s_2^2 \neq s_3^2 = s_4^2 \\ H_1 + x_3 H_2 + H_3 + x_3 H_4 & s_1^2 = s_2^2 \neq s_3^2 = s_4^2 \\ H_1 + H_2 + x_3 H_3 + x_3^2 H_4 & s_1^2 \neq s_2^2 = s_3^2 = s_4^2 \\ H_1 + x_3 H_2 + x_3^2 H_3 + x_3^3 H_4 & s_1^2 = s_2^2 = s_3^2 = s_4^2 \end{cases}$$

where H_q satisfy the following second-order equations

$$\nabla_Q^2 H_q = 0 \tag{11}$$

in which the upper case subscript Q takes the same number as the corresponding lower case q, but with no summation convention. Therefore, the eighth-order Eq. (8) has been replaced with four quasi-harmonic equations.

In this paper, only the case of distinct values s_q is concerned. Then, the general solution can be written as

$$u_1 = \lambda_{1q}\partial_1\partial_3^4 H_q, u_3 = \lambda_{2q}\partial_3^5 H_q,$$
$$w_1 = \lambda_{3q}\partial_1\partial_3^4 H_q, T = \lambda_{4q}\partial_3^6 H_q \tag{12}$$

Here set $\alpha = 1,2,3$, where

$$\lambda_{\alpha q} = a_\alpha - b_\alpha \frac{1}{s_q^2} + c_\alpha \frac{1}{s_q^4}, \lambda_{4q} = a - b\frac{1}{s_q^2} + c\frac{1}{s_q^4} - d\frac{1}{s_q^6}$$

For further simplification, assume that

$$\psi_q = \lambda_{1Q}\partial_3^4 H_q \tag{13}$$

From Eqs. (8,11), it can be seen that ψ_q satisfy the following equations

$$\nabla_Q^2 \psi_q = 0 \tag{14}$$

Therefore, the general solutions of the thermoelastic plane problems of 2D decagonal QCs can be expressed in terms of the four quasi-harmonic functions ψ_q as follows

$$\begin{cases} u_1 = \delta_{Qq}\partial_1\psi_q, u_3 = m_{1q}\partial_3\psi_q \\ w_1 = m_{2q}\partial_1\psi_q, T = m_{3q}\partial_3^2\psi_q \end{cases} \tag{15}$$

$$\begin{cases} \sigma_{11} = -l_{1q}\partial_3^2\psi_q, \sigma_{33} = l_{1q}\partial_3^2\psi_q/s_Q^2, \sigma_{13} = l_{1q}\partial_1\partial_3\psi_q \\ H_{11} = -l_{2q}\partial_3^2\psi_q, H_{13} = l_{2q}\partial_1\partial_3\psi_q \\ l_{1q} = C_{44}(\delta_{Qq} + m_{1q}), l_{2q} = K_4 m_{2q} \end{cases} \tag{16}$$

where δ_{Qq} is the Kronecker Delta symbol, $m_{1q} = \lambda_{2q}/\lambda_{1Q}$, $m_{2q} = \lambda_{3q}/\lambda_{1Q}$, $m_{3q} = \lambda_{4q}/\lambda_{1Q}$. When $q=1$, 2, 3, $m_{3q}=0$.

Eqs. (15-16) are the general solutions of this thermoelastic plane problem in terms of displacement functions ψ_q. If boundary conditions are given, the analytic solutions can be obtained for the boundary value problems.

4 Steady Point Heat Source in Semi-infinite Plane

Consider a semi-infinite QC plane $x_3 \geqslant 0$ whose quasi-periodic direction is the x_1-axis, and whose periodic direction is x_3-axis as in Fig. 1. A point heat source H is applied at the point $(0, h)$ in 2D Cartesian coordinates (x_1, x_3) and the surface $(x_3 = 0)$ is free. Based on the general solutions, the thermal field in the semi-infinite QC plane is derived in this section.

The boundary conditions on the surface $x_3 = 0$ are

$$\sigma_{33} = \sigma_{13} = 0, H_{13} = 0, \partial_3 T = 0 \tag{17}$$

For future reference, a series of denotations are introduced as follows

$$z_q = s_q x_3, h_k = s_k h$$
$$z_{qk} = z_q + h_k, r_{qk} = \sqrt{x_1^2 + z_{qk}^2}$$
$$\bar{z}_{qk} = z_q - h_k, \bar{r}_{qk} = \sqrt{x_1^2 + \bar{z}_{qk}^2}$$

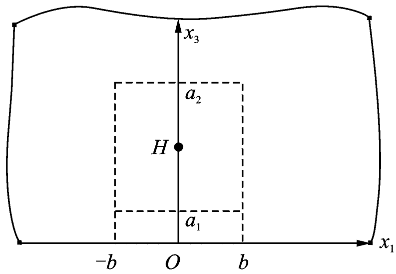

Fig. 1 Semi-infinite QC plane applied by a point source H

Introduce the following harmonic functions

$$\psi_q = A_q \cdot \left[\frac{1}{2}(\bar{z}_{QQ}^2 - x_1^2)\left(\ln\bar{r}_{QQ} - \frac{3}{2}\right) - x_1\bar{z}_{QQ}\arctan\frac{x_1}{\bar{z}_{QQ}}\right] + A_{qk}\left[\frac{1}{2}(z_{Qk}^2 - x_1^2)\left(\ln r_{Qk} - \frac{3}{2}\right) - x_1 z_{Qk}\arctan\frac{x_1}{z_{Qk}}\right] \tag{18}$$

where A_q and A_{qk} are twenty constants to be determined. Substituting Eq. (18) into Eqs. (15—16), the expressions of the coupled field are as follows

$$\begin{cases} u_1 = -\delta_{Qq}A_q[x_1(\ln\bar{r}_{QQ}-1)+\bar{z}_{QQ}\arctan(x_1/\bar{z}_{QQ})] - \\ \quad \delta_{Qq}A_{qk}[x_1(\ln r_{Qk}-1)+z_{Qk}\arctan(x_1/z_{Qk})] \\ u_3 = m_{1q}s_qA_Q[\bar{z}_{QQ}(\ln\bar{r}_{QQ}-1)+x_1\arctan(x_1/\bar{z}_{QQ})] + \\ \quad m_{1q}s_qA_{Qk}[z_{Qk}(\ln r_{Qk}-1)+x_1\arctan(x_1/z_{Qk})] \\ w_1 = -m_{2q}A_q[x_1(\ln\bar{r}_{QQ}-1)+\bar{z}_{QQ}\arctan(x_1/\bar{z}_{QQ})] - \\ \quad m_{2q}A_{qk}[x_1(\ln r_{Qk}-1)+z_{Qk}\arctan(x_1/z_{Qk})] \\ T = m_{34}A_4\ln\bar{r}_{44} + m_{34}A_{4k}\ln\bar{r}_{4k} \end{cases} \tag{19}$$

$$\begin{cases} \sigma_{11} = -s_q^2 l_{1q}A_Q\ln\bar{r}_{QQ} - s_q^2 l_{1q}A_{Qk}\ln r_{Qk} \\ \sigma_{33} = l_{1q}A_q\ln\bar{r}_{QQ} + l_{1q}A_{qk}\ln r_{Qk} \\ \sigma_{13} = -s_q l_{1q}A_Q\arctan(x_1/\bar{z}_{QQ}) - s_q l_{1q}A_{Qk}\cdot \\ \quad \arctan(x_1/z_{Qk}) \\ H_{11} = -s_q^2 l_{2q}A_Q\ln\bar{r}_{QQ} - s_q^2 l_{2q}A_{Qk}\ln r_{Qk} \\ H_{13} = -s_q l_{2q}A_Q\arctan(x_1/\bar{z}_{QQ}) - s_q l_{2q}A_{Qk}\cdot \\ \quad \arctan(x_1/z_{Qk}) \end{cases} \tag{20}$$

Considerating the continuity on plane $x_3 = h$ for u_3, σ_{13} and H_{13} yields

$$m_{1q}s_qA_Q = 0 \tag{21}$$

$$l_{\beta q}s_qA_Q = 0 \quad \beta = 1,2 \tag{22}$$

Substituting Eq. (16) into Eq. (22), by virtue of Eq. (21), Eq. (22) can be simplified into one equation

$$s_qA_q = 0, m_{\beta q}s_qA_Q = 0 \tag{23}$$

When the phonon, phason and thermal equilibrium for a cylinder of $a_1 \leqslant x_3 \leqslant a_2$ $(0 < a_1 < h < a_2)$ and $0 \leqslant r \leqslant b$ are considered, three additional equations can be obtained

$$\int_{-b}^{b}[\sigma_{33}(x_1,a_2) - \sigma_{33}(x_1,a_1)]\,\mathrm{d}x_1 + \int_{a_1}^{a_2}[\sigma_{13}(b,x_3) - \sigma_{13}(-b,x_3)]\,\mathrm{d}x_3 = 0 \tag{24}$$

$$-k_{33}\int_{-b}^{b}[\partial_3T(x_1,a_2) - \partial_3T(x_1,a_1)]\,\mathrm{d}x_1 - k_{11}\int_{a_1}^{a_2}[\partial_3T(b,x_3) - \partial_3T(-b,x_3)]\,\mathrm{d}x_3 = H \tag{25}$$

Substituting Eq. (20) into Eqs. (24—25) and integrating, we can obtain

$$l_{1q}A_qI_1 + l_{1q}\delta_{Kk}A_{qk}I_2 = 0 \tag{26}$$

$$A_4I_3 + \delta_{Kk}A_{4k}I_4 = H/m_{34}\sqrt{\mu_{11}\mu_{33}} \tag{27}$$

where

$$I_1 = I_2 = I_4 = 0, I_3 = -2\pi \tag{28}$$

Thus, Eq. (24) is satisfied automatically. A_4 can be determined by Eq. (27) as follows

$$A_4 = -\frac{H}{2\pi m_{34}\sqrt{k_{11}k_{33}}} \tag{29}$$

Finally, when the coupled field on the surface $x_3 = 0$ is considered, substituting general solutions Eqs. (15—16) into boundary conditions Eq. (17) and using $s_4 = \sqrt{k_{11}/k_{33}}$, we can obtain

$$\begin{aligned} -s_ql_{\beta Q}A_Q + s_kl_{\beta k}A_{qK} &= 0 \\ l_{1q}A_Q + l_{1k}A_{qk} &= 0 \\ A_4 - A_{44} &= 0 \\ A_{4\alpha} &= 0 \end{aligned} \tag{30}$$

Thus, twenty constants A_q and A_{qk} can be determined by twenty equations, including Eqs. (23, 29—30).

5 Conclusions

On the basis of the operator method and the introduction of the displacement function H, the general solutions of thermoelastic plane problems of 2D decagonal QCs are first presented. The introduced displacement function H has to satisfy an eighth-order partial differential equation. Owing to complexity of the higher order equation, it is difficult to obtain rigorous analytic solutions directly and not applicable in most cases. Based on the Almansi's theorem, and by virtue of a decomposition and superposition procedure, the general solution is further simplified in terms of four quasi-harmonic functions ψ_q. Considering that the characteristic roots s_q^2 are distinct, the obtained general solutions of 2D decagonal QCs are in simple forms which are conveniently applied. As an application of the general solution, for a steady point heat source in a semi-infinite QC plane, the closed form solutions are presented by the four harmonic functions.

The general solutions are very convenient to be used to study the inhomogeneity and defect problems of 2D decagonal QCs. These also provide basis to judge the rationality of the solutions by the finite element method or the boundary element method. The analysis method in this paper can also be used to solve the more complicated thermoelastic plane problems of 2D QCs.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (11172319); the Chinese Universities Scientific Fund (2011JS046, 2013BH008); the Opening Fund of State Key Laboratory of Nonlinear Mechanics; the Program for New Century Excellent Talents in University (NCET-13-0552); the National Science Foundation for Post-doctoral Scientists of China (2013M541086).

References:

[1] Hu Chengzheng, Yang Wenge, Wang Renhui, et al. Quasicrystal symmetry and physical properties[J]. Progress in Physics, 1997, 17(4): 345-375. (in Chinese)

[2] Wang, X, Zhang J Q. A steady line heat source in a decagonal quasicrystalline half-space[J]. Mechanics Research Communications, 2005, 32(4): 420-428.

[3] Li X Y, Li P D. Three-dimensional thermo-elastic general solutions of one-dimensional hexagonal quasi-crystal and fundamental solutions[J]. Physics Letters A, 2012, 376(26/27): 2004-2009.

[4] Xiong S M, Hou P F, Yang S Y. 2-D green's functions for semi-infinite orthotropic piezothermoelastic plane[J]. Ultrasonics, Ferroelectrics and Frequency Control, 2010, 57(5): 1003-1010.

[5] Kumar R, Chawla V. General steady-state solution and green's function in orthotropic piezothermoelastic diffusion medium[J]. Archives of Mechanics, 2012, 64(6): 555-579.

[6] Gao Y, Zhao B S. A general treatment of three-dimensional elasticity of quasicrystals by an operator method[J]. Physical Status Solidi (b), 2006, 243 (15): 4007-4019.

[7] Hu H. On the three-dimensional problems of the theory of elasticity of a transversely isotropic body [J]. Scientia Sinica, 1953, 9(2): 145-151.

[8] Ding H J, Chen B, Liang J. General solutions for coupled equations for piezoelectric media[J]. International Journal of Solids and Structures, 1996, 33 (16): 2283-2298.

[9] Wang M Z, Wang W. Completeness and nonuniqueness of general solutions of transversely isotropic elasticity[J]. International Journal of Solids and Structures, 1995, 32(3): 501-513.

[10] Wang W, Shi M X. On the general solutions of transversely isotropic elasticity [J]. International Journal of Solids and Structures, 1998, 35(25): 3283-3297.

[11] Ding H J, Cheng B, Liang J. On the Green's functions for two-phase transversely isotropic piezoelectric media[J]. International Journal of Solids and Structures, 1997, 34(23): 3041-3057.

(Executive editor: Xu Chengting)