Problem of Circular Hole in Thermopiezoelectric Media with Semi-permeable Thermal Boundary Condition

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Abstract: The semi-permeable boundary condition is proposed to discuss the influence of the thermal conductivity acting on the stress and heat flow around the hole. Based on the Stroh formalism, the closed form solutions are derived, the stress and heat flow around the hole are discussed. The results show that the thermal boundary condition has significant influence on the hoop stress and heat flow around the hole. The hoop stress decreases dramatically with the increasement of the thermal conduction coefficient.

Key words: thermopiezoelectric; thermal conductivity; hoop stress; heat flow

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1 Introduction

Widespread attention has been given to the thermal stress problems with inclusions, holes or cracks. For example, Florence and Goodier[1], Sih[2], Parton[3], Zhang and Hasebe[4], Chao and Shen[5-6], Kattis and Patia[7], and Kaminskii and Flegantov[8] studied the thermal stress problems in isotropic media, and also Sturla and Barber[9], Hwu[10-11], Tarn and Wang[12], Chao and Chang[13], Lin et al[14] and Shen and Kuang[15] discussed the thermal stress problems in anisotropic materials.

In recent years, the thermo-electric-mechanical coupling problem in thermopiezoelectric media with holes or cracks has also received much attention with increasingly wide application of thermopiezoelectric materials in the engineering. Gao and Wang[16] studied the 2D problem of thermopiezoelectric materials with cracks by means of the Parton assumption, i. e. the crack is considered as a thin slit and thus the normal components of electric displacement and the tangential component of electric field are assumed to be continuous across the slit[3]. They also presented an exact solution for the problem of an elliptic hole or a crack in a thermopiezoelectric solid[17]. The fracture analysis of a cracked thermopiezoelectric medium with thermoelectric loading has been dealt with by Ueda[18].

However, all the references above supposed that the normal component of the heat flow could be treated as zero at the rim of the hole. In the present work, a semi-permeable thermal boundary condition is proposed to discuss the influence of the thermal conductivity acting on the stress and heat flow around the hole.

2 Basic Equations

The governing equations for piezothermoelastic problem can be expressed, in the stationary case without body force, extrinsic bulk charge and heat source, as folllows:

Governing equation

$$q_{i,i}=0 \tag{1}$$

Constitutive equation

$$q_i=-\lambda_{ij}T_{,j} \tag{2}$$

where q_i, λ_{ij}, $T\,(i,j=1,2)$ are the heat flux, heat conduction coefficients and temperature, respectively.

From Eqs. (1—2), we have

$$T=2\mathrm{Re}[g'(z_t)] \quad z_t=x_1+\mu_t x_2 \tag{3}$$

where $\mu_t=(-\lambda_{12}+\mathrm{i}\kappa_t)/\lambda_{22}$, $\kappa_t=(\lambda_{11}\lambda_{22}-\lambda_{12}^2)^{1/2}$,

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$\lambda_{11}\lambda_{22}-\lambda_{12}^2>0$.

Substituting Eq. (3) into Eq. (2) yields

$$q_1=2\mathrm{Re}[\mathrm{i}\mu_t\kappa_t g''(z_t)] \tag{4}$$

$$q_2=-2\mathrm{Re}[\mathrm{i}\kappa_t g''(z_t)] \tag{5}$$

and

$$q_n\mathrm{d}s=q_1\mathrm{d}x_2-q_2\mathrm{d}x_1 \tag{6}$$

On the other hand, the resultant heat flow Q can be expressed as

$$Q=\int q_n\mathrm{d}s \tag{7}$$

Substituting Eqs. (4—6) into Eq. (7) yields

$$Q=2\mathrm{Re}[\mathrm{i}\kappa_t g'(z_t)] \tag{8}$$

The semi-permeable boundary conditions of heat flow is

$$q_n=\lambda_1 q_1^\infty\cos\theta+\lambda_2 q_2^\infty\sin\theta \tag{9}$$

where λ_1 and λ_2 are two thermal conduction coefficients.

The complete set of governing equations are

$$\sigma_{ij}=c_{ijkl}\gamma_{kl}-e_{ijs}E_s-\beta_{ij}T \tag{10}$$

$$D_i=\varepsilon_{is}E_s+e_{irs}\gamma_{rs}+\tau_i T \tag{11}$$

where $c_{ijkl}=c_{jikl}=c_{ijlk}=c_{klij}$; $e_{kij}=e_{kji}$, $\varepsilon_{ij}=\varepsilon_{ji}$; $\beta_{ij}=\beta_{ji}$; $i,j,k,l=1,2,3,\cdots$. σ, $\boldsymbol{D}$, $\boldsymbol{E}$ are stress, electric displacement and electric field, respectively; c_{ijkl}, e_{kij}, ε_{ij}, β_{ij} and τ_i the elasticity constants, piezoelectricity constants, dielectric constants, stress-temperature coefficients and pyroelectric coefficients, respectively.

Equilibrium equation are

$$\sigma_{ij,j}=0,D_{i,i}=0 \tag{12}$$

Substituting Eqs. (10—11) into Eq. (12) yields

$$(c_{ijrs}u_r+e_{sji}\varphi)_{,si}-\beta_{ij}T_{,i}=0$$
$$(-\varepsilon_{is}\varphi+e_{irs}u_r)_{,si}+\tau_i T_{,i}=0 \tag{13}$$

Introduce two function vectors $\boldsymbol{u}=(u_1,u_2,u_3,\phi)^{\mathrm{T}}$ and $\boldsymbol{\varphi}=(\varphi_1,\varphi_2,\varphi_3,\varphi_4)^{\mathrm{T}}$。

The homogeneous solutions of Eq. (13) are

$$\begin{aligned}\boldsymbol{u}_h&=\boldsymbol{A}\boldsymbol{f}(z_*)+\overline{\boldsymbol{A}}\overline{\boldsymbol{f}(z_*)}\\ \boldsymbol{\varphi}_h&=\boldsymbol{B}\boldsymbol{f}(z_*)+\overline{\boldsymbol{B}}\overline{\boldsymbol{f}(z_*)}\end{aligned} \tag{14}$$

and the particular solutions are

$$\begin{aligned}\boldsymbol{u}_p&=2\mathrm{Re}[\boldsymbol{c}g(z_t)]\\ \boldsymbol{\varphi}_p&=2\mathrm{Re}[\boldsymbol{d}g(z_t)]\end{aligned} \tag{15}$$

where $\boldsymbol{c}$ and $\boldsymbol{d}$ are the heat eigenvectors, which can be determined from the following equations

$$\boldsymbol{D}_*(\mu_t)\boldsymbol{c}=\boldsymbol{\beta}_1+\mu_t\boldsymbol{\beta}_2$$
$$\boldsymbol{d}=(\boldsymbol{R}^{\mathrm{T}}+\mu_t\boldsymbol{W})\boldsymbol{c}-\boldsymbol{\beta}_2$$

$$\begin{aligned}\boldsymbol{\beta}_1&=(\beta_{11},\beta_{21},\beta_{31},\tau_1)^{\mathrm{T}}\\ \boldsymbol{\beta}_2&=(\beta_{12},\beta_{22},\beta_{32},\tau_2)^{\mathrm{T}}\end{aligned} \tag{16}$$

The final solutions of $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ are

$$\begin{aligned}\boldsymbol{u}&=2\mathrm{Re}[\boldsymbol{A}\boldsymbol{f}(z)+\boldsymbol{c}g(z)]\\ \boldsymbol{\varphi}&=2\mathrm{Re}[\boldsymbol{B}\boldsymbol{f}(z)+\boldsymbol{d}g(z)]\end{aligned} \tag{17}$$

And $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy the following orthogonality relation

$$\begin{bmatrix}\boldsymbol{B}^{\mathrm{T}} & \boldsymbol{A}^{\mathrm{T}}\\ \overline{\boldsymbol{B}}^{\mathrm{T}} & \overline{\boldsymbol{A}}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{A} & \overline{\boldsymbol{A}}\\ \boldsymbol{B} & \overline{\boldsymbol{B}}\end{bmatrix}=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix} \tag{18}$$

Assuming that the considered problem satisfies such a condition that for an arbitrary point on the boundary, the corresponding points z_t and $z_\alpha(\alpha=1-4)$ can be translated into an identical point, e. g. on the x_1-axis or an unit circle, and as a result the boundary equation can be reduced to that containing one variable. Only under this condition, the one-complex-variable approach introduced by Suo[19] can be used to simplify analysis when the boundary conditions are considered[16,20]. In the present work these one-complex variable equations can be summarized as

$$T=2\mathrm{Re}[g'(z)] \tag{19}$$

$$q_1=2\mathrm{Re}[\mathrm{i}\mu_t\kappa_t g''(z)] \tag{20}$$

$$q_2=-2\mathrm{Re}[\mathrm{i}\kappa_t g''(z)] \tag{21}$$

$$Q=2\mathrm{Re}[\mathrm{i}g'(z)]=\frac{1}{\kappa_t}\int q_1\mathrm{d}x_2-q_2\mathrm{d}x_1 \tag{22}$$

Consider a generalized 2D problem of a thermopiezoelectric medium containing an circular hole as shown Fig. 1.

The boundary conditions at the rim of the hole are

$$2\mathrm{Re}[\boldsymbol{B}\boldsymbol{f}(z)+\boldsymbol{d}g(z)]=\int_s \boldsymbol{t}\mathrm{d}s$$

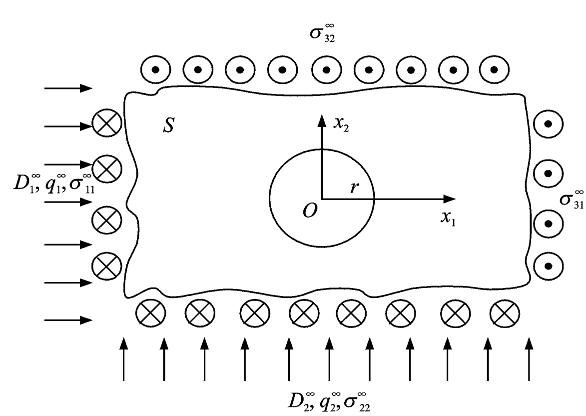

Fig. 1 Circular hole in thermopiezoelectric solid

$$\boldsymbol{t}=[t_1,t_2,t_3,D_n]^{\mathrm{T}} \tag{23}$$

3 Temperature Field in Medium

From Eq. (9), the semi-permeable boundary conditions of heat flow is

$$-2\mathrm{Re}[\mathrm{i}\kappa_t g''(z)]\sin\theta+2\mathrm{Re}[\mathrm{i}\mu_t\kappa_t g''(z)]\cos\theta= \lambda_1 q_1\cos\theta+\lambda_2 q_2\sin\theta \tag{24}$$

$g'(z)$ takes the form of

$$g'(z)=c_t^{(2)}z+g'_0(z) \tag{25}$$

where $g'_0(z)$ is a holomorphic function outside the hole, $g'_0(\infty)=c_t^{(1)}$, $c_t^{(1)}$ is a constant corresponding to an uniform temperature field and thus can be neglected without loss in generality, and $c_t^{(2)}$ is another constant to be determined.

Substituting Eq. (25) into Eqs. (20—21), and taking the limiting $z\rightarrow\infty$ yields

$$2\mathrm{Re}[\mathrm{i}\mu_t\kappa_t c_t^{(2)}]=q_1^{\infty} \tag{26}$$

$$2\mathrm{Re}[\mathrm{i}\kappa_t c_t^{(2)}]=-q_2^{\infty} \tag{27}$$

Eqs. (26—27) give

$$c_t^{(2)}=\frac{q_1^{\infty}+\bar{\mu}_t q_2^{\infty}}{\mathrm{i}\kappa_t(\mu_t-\bar{\mu}_t)} \tag{28}$$

The following transform functions

$$z_\alpha(\zeta)=R_\alpha(\zeta+m_\alpha\zeta^{-1}) \tag{29}$$

where $R_\alpha=\frac{r}{2}(1-\mathrm{i}\mu_\alpha)$, $m_\alpha=\frac{1+\mathrm{i}\mu_\alpha}{1-\mathrm{i}\mu_\alpha}$ $(\alpha=1-4)$. Eq. (29) maps the ellipse in the z_α-plane into a unit circle in the ζ plane.

Noting that on the hole, $\zeta=\sigma=\mathrm{e}^{\mathrm{i}\theta}$, and

$$\cos\theta=\frac{1}{2}\left(\frac{1}{\sigma}+\sigma\right) \tag{30}$$

$$\sin\theta=\frac{\mathrm{i}}{2}\left(\frac{1}{\sigma}-\sigma\right) \tag{31}$$

Eq. (24) can be rewriten as

$$-2\mathrm{Re}[\mathrm{i}\kappa_t g''_0(z)]\frac{\mathrm{i}}{2}\left(\frac{1}{\sigma}-\sigma\right)+ 2\mathrm{Re}[\mathrm{i}\mu_t\kappa_t g''_0(z)]\frac{1}{2}\left(\frac{1}{\sigma}+\sigma\right)= (\lambda_1-1)q_1^{\infty}\frac{1}{2}\left(\frac{1}{\sigma}+\sigma\right)+ (\lambda_2-1)q_2^{\infty}\frac{\mathrm{i}}{2}\left(\frac{1}{\sigma}-\sigma\right) \tag{32}$$

Calculating the Cauchy integration leads to

$$g''_0(z)=\frac{\mathrm{i}}{\kappa_t}\left[(1-\lambda_2)q_2^{\infty}+\frac{(1-\lambda_1)q_1^{\infty}}{\mu_t}\right]\frac{1}{\zeta^2} \tag{33}$$

Therefore the final form of $g''(z)$ can be expressed as

$$g''(z)=c_t^{(2)}+\frac{\mathrm{i}}{\kappa_t}\left[(1-\lambda_2)q_2^{\infty}+\frac{(1-\lambda_1)q_1^{\infty}}{\mu_t}\right]\frac{1}{\zeta^2} \tag{34}$$

The integration of Eq. (34) with respect to z gives

$$g'(z)=c_t^{(2)}z+c_t^{(1)}-\frac{\mathrm{i}}{\kappa_t}\left[(1-\lambda_2)q_2^{\infty}+\frac{(1-\lambda_1)q_1^{\infty}}{\mu_t}\right]\frac{R}{\zeta}+\frac{\mathrm{i}}{\kappa_t}\left[(1-\lambda_2)q_2^{\infty}+\frac{(1-\lambda_1)q_1^{\infty}}{\mu_t}\right]\frac{Rm}{\zeta^3} \tag{35}$$

$$g(z)=\frac{1}{2}c_t^{(2)}z^2+c_t^{(1)}z-\frac{\mathrm{i}}{\kappa_t}\left[(1-\lambda_2)q_2^{\infty}+\frac{(1-\lambda_1)q_1^{\infty}}{\mu_t}\right]R^2\ln\zeta-\frac{\mathrm{i}}{\kappa_t}\left[(1-\lambda_2)q_2^{\infty}+\frac{(1-\lambda_1)q_1^{\infty}}{\mu_t}\right]\frac{R^2m}{\zeta^2}+\frac{\mathrm{i}}{\kappa_t}\left[(1-\lambda_2)q_2^{\infty}+\frac{(1-\lambda_1)q_1^{\infty}}{\mu_t}\right]\frac{R^2m^2}{4\zeta^4} \tag{36}$$

4 Electro-Elastic Field in Medium

Observing Eq. (36), the complex potential in the medium can be expressed as

$$\boldsymbol{f}(z)=\frac{1}{2}\boldsymbol{c}^{(2)}z^2+\boldsymbol{c}^{(1)}z+\boldsymbol{\delta}\ln\zeta(z)+\boldsymbol{f}_0(z) \tag{37}$$

where $\boldsymbol{f}_0(z)$ is a holomorphic function outside the hole; $\boldsymbol{c}^{(2)}$, $\boldsymbol{c}^{(1)}$ and $\boldsymbol{\delta}$ are the three constant vectors to be found.

The force equilibrium condition and the conditions of single-valued displacement and electric potential require

$$\oint_{\Gamma_n}\boldsymbol{u}_{,1}\mathrm{d}z=0,\ \oint_{\Gamma_n}\boldsymbol{\varphi}_{,1}\mathrm{d}z=0 \tag{38}$$

where Γ_n stands for a clockwise closed-contour encircling the hole, and

$$\boldsymbol{u}_{,1}=2\mathrm{Re}[\boldsymbol{A}\boldsymbol{f}'(z)+\boldsymbol{c}g'(z)] \tag{39}$$

$$\boldsymbol{\varphi}_{,1}=2\mathrm{Re}[\boldsymbol{A}\boldsymbol{f}'(z)+\boldsymbol{c}g'(z)] \tag{40}$$

Substituting Eq. (37) into Eqs. (39—40), and then using the residue theorem produces

$$[\boldsymbol{A}\delta+\boldsymbol{c}\gamma]-\overline{[\boldsymbol{A}\delta+\boldsymbol{c}\gamma]}=0 \tag{41}$$

$$[\boldsymbol{B}\delta+\boldsymbol{d}\gamma]-\overline{[\boldsymbol{B}\delta+\boldsymbol{d}\gamma]}=0 \tag{42}$$

where $\gamma=-\frac{\mathrm{i}R^2}{\kappa_t}\left[(1-\lambda_2)q_2^{\infty}+\frac{(1-\lambda_1)q_1^{\infty}}{\mu_t}\right]$.

Using Eq. (18) and Eqs. (41—42) yields

$$\delta = \boldsymbol{B}^{\mathrm{T}}(\overline{\boldsymbol{c}\gamma} - \boldsymbol{c}\gamma) + \boldsymbol{A}^{\mathrm{T}}(\overline{\boldsymbol{d}\gamma} - \boldsymbol{d}\gamma) \quad (43)$$

Considering the fact that both the stresses and strains are bounded at infinity, we have

$$2\mathrm{Re}[\boldsymbol{A}\boldsymbol{c}^{(1)} + \boldsymbol{c}c_t^{(1)}] = \boldsymbol{u}_{,1}^{\infty} \quad (44)$$

$$2\mathrm{Re}[\boldsymbol{B}\boldsymbol{c}^{(1)} + \boldsymbol{d}c_t^{(1)}] = \boldsymbol{\varphi}_{,1}^{\infty} \quad (45)$$

$$2\mathrm{Re}[(\boldsymbol{A}\boldsymbol{c}^{(2)} + \boldsymbol{c}c_t^{(2)})z] = 0 \quad (46)$$

$$2\mathrm{Re}[(\boldsymbol{B}\boldsymbol{c}^{(2)} + \boldsymbol{d}c_t^{(2)})z] = 0 \quad (47)$$

Using Eq. (18) one obtains from Eqs. (44—45) that

$$\boldsymbol{c}^{(1)} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{u}_{,1}^{\infty} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\varphi}_{,1}^{\infty} - \boldsymbol{B}^{\mathrm{T}}[\boldsymbol{c}c_t^{(2)} + \overline{\boldsymbol{c}c_t^{(2)}}] - \boldsymbol{A}^{\mathrm{T}}[\boldsymbol{d}c_t^{(2)} + \overline{\boldsymbol{d}c_t^{(2)}}] \quad (48)$$

On the other hand, Eqs. (46—47) imply that the complex functions $(\boldsymbol{A}\boldsymbol{c}^{(2)} + \boldsymbol{c}c_t^{(2)})z$ and $(\boldsymbol{B}\boldsymbol{c}^{(2)} + \boldsymbol{d}c_t^{(2)})z$, which are corresponding to the uniform heat flow in an infinite medium without holes, will not produce stress and strain, and thus can be cut out in the boundary equations. Therefore $\boldsymbol{\varphi}$ and $\boldsymbol{u}$ can be rewriten as

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^{\infty} + 2\mathrm{Re}[(\boldsymbol{B}\delta + \boldsymbol{d}\gamma)\ln\zeta] + 2\mathrm{Re}\left[\boldsymbol{B}\boldsymbol{f}_0(z) + \boldsymbol{d}\left(m\gamma\zeta^{-2} - \frac{1}{4}m^2\gamma\zeta^{-4}\right)\right] \quad (49)$$

$$\boldsymbol{u} = \boldsymbol{u}^{\infty} + 2\mathrm{Re}[(\boldsymbol{A}\delta + \boldsymbol{c}\gamma)\ln\zeta] + 2\mathrm{Re}\left[\boldsymbol{A}\boldsymbol{f}_0(z) + \boldsymbol{c}\left(m\gamma\zeta^{-2} - \frac{1}{4}m^2\gamma\zeta^{-4}\right)\right] \quad (50)$$

where

$$\boldsymbol{\varphi}^{\infty} = \boldsymbol{\Pi}_2^{\infty}x_1 - \boldsymbol{\Pi}_1^{\infty}x_2,\ \boldsymbol{u}^{\infty} = \boldsymbol{\varepsilon}_1^{\infty}x_1 + \boldsymbol{\varepsilon}_2^{\infty}x_2$$

$$\boldsymbol{\Pi}_1^{\infty} = (\sigma_{11}^{\infty}, \sigma_{12}^{\infty}, \sigma_{13}^{\infty}, D_1^{\infty})^{\mathrm{T}} = -\boldsymbol{\varphi}_{,2}^{\infty}$$

$$\boldsymbol{\Pi}_2^{\infty} = (\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}, D_2^{\infty})^{\mathrm{T}} = -\boldsymbol{\varphi}_{,1}^{\infty}$$

$$\boldsymbol{\varepsilon}_1^{\infty} = (\varepsilon_{11}^{\infty}, \varepsilon_{12}^{\infty} + \omega_3^{\infty}, 2\varepsilon_{13}^{\infty}, -E_1^{\infty})^{\mathrm{T}} = \boldsymbol{u}_{,1}^{\infty}$$

$$\boldsymbol{\varepsilon}_2^{\infty} = (\varepsilon_{21}^{\infty} - \omega_3^{\infty}, \varepsilon_{22}^{\infty}, 2\varepsilon_{23}^{\infty}, -E_2^{\infty})^{\mathrm{T}} = \boldsymbol{u}_{,2}^{\infty}$$

On the hole, we have

$$\boldsymbol{\varphi}(\sigma) = \boldsymbol{\Pi}_2^{\infty}x_1(\sigma) - \boldsymbol{\Pi}_1^{\infty}x_2(\sigma) + 2\mathrm{Re}[\boldsymbol{K}_0(\sigma)] \quad (51)$$

$$\boldsymbol{u}(\sigma) = \boldsymbol{\varepsilon}_1^{\infty}x_1(\sigma) + \boldsymbol{\varepsilon}_2^{\infty}x_2(\sigma) + 2\mathrm{Im}\left[\boldsymbol{Y}\boldsymbol{K}_0(\sigma) - \boldsymbol{M}\left(m\gamma\sigma^{-2} - \frac{1}{4}m^2\gamma\sigma^{-4}\right)\right] \quad (52)$$

where

$$\boldsymbol{K}_0(\zeta) = \boldsymbol{B}\boldsymbol{f}_0(z) + \boldsymbol{d}\left(m\gamma\zeta^{-2} - \frac{1}{4}m^2\gamma\zeta^{-4}\right)$$

$$\boldsymbol{Y} = \mathrm{i}\boldsymbol{A}\boldsymbol{B}^{-1},\ \boldsymbol{M} = \boldsymbol{Y}\boldsymbol{d} - \mathrm{i}\boldsymbol{c}$$

Ignore the electric field within the hole, the boundary condition is

$$\boldsymbol{\varphi}(\sigma) = (0,0,0,0)^{\mathrm{T}} \quad (53)$$

Namely

$$\boldsymbol{\Pi}_2^{\infty}x_1(\sigma) - \boldsymbol{\Pi}_1^{\infty}x_2(\sigma) + 2\mathrm{Re}[\boldsymbol{K}_0(\sigma)] = (0,0,0,0)^{\mathrm{T}} \quad (54)$$

One can obtain after calculating the Cauchy integration that

$$\boldsymbol{K}_0(\zeta) = -\frac{r}{2}(\boldsymbol{\Pi}_2^{\infty} - \mathrm{i}\boldsymbol{\Pi}_1^{\infty})\zeta^{-1} \quad (55)$$

So far, all the field variables can be calculated.

5 Stresses on Hole Rim

The stress components are

$$\sigma_{11} = -\varphi_{1,2},\ \sigma_{12} = \varphi_{1,1},\ \sigma_{22} = \varphi_{2,1} \quad (56)$$

$$\sigma_{\varphi\varphi} = \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\sigma_{11} - \sigma_{22}}{2}\cos 2\theta - \sigma_{12}\sin 2\theta \quad (57)$$

Consider a transversly isotropic piezoelectric medium cadmium selenide, where the poling direction is parallel to the X_3-axis. The material constants are

$c_{11} = 74.1\times10^9\ \mathrm{N\cdot m^{-2}}$, $c_{12} = 45.2\times10^9\ \mathrm{N\cdot m^{-2}}$

$c_{13} = 39.3\times10^9\ \mathrm{N\cdot m^{-2}}$, $c_{33} = 83.6\times10^9\ \mathrm{N\cdot m^{-2}}$

$c_{44} = 13.2\times10^9\ \mathrm{N\cdot m^{-2}}$, $e_{31} = -0.16\ \mathrm{C\cdot m^{-2}}$

$e_{33} = 0.347\ \mathrm{C\cdot m^{-2}}$, $e_{15} = -0.138\ \mathrm{C\cdot m^{-2}}$

$\varepsilon_{11} = 82.6\times10^{-12}\ \mathrm{C^2\cdot N^{-1}\cdot m^{-2}}$

$\varepsilon_{33} = 90.3\times10^{-12}\ \mathrm{C^2\cdot N^{-1}\cdot m^{-2}}$

$\beta_{11} = 0.621\times10^6\ \mathrm{N\cdot K^{-1}\cdot m^{-2}}$

$\beta_{33} = 0.551\times10^6\ \mathrm{N\cdot K^{-1}\cdot m^{-2}}$

$\tau_3 = -2.94\times10^{-6}\ \mathrm{C\cdot K^{-1}\cdot m^{-2}}$

$\lambda_{11} = \lambda_{22} = 10^{-6}\ \mathrm{W\cdot m^{-1}\cdot K^{-1}}$, $\lambda_{33} = 1.5\lambda_{11}$

If our attention is focused on the field in X_1-X_3 plane, the out-of plane displacement does not couple with the in-plane displacements and the electric potential, and the elastic matrices $\boldsymbol{S}$, $\boldsymbol{R}$ and $\boldsymbol{W}$ degenerate into the 3×3 ones

$$\boldsymbol{S} = \begin{bmatrix} c_{11} & 0 & 0 \\ 0 & c_{44} & e_{15} \\ 0 & e_{15} & -\varepsilon_{11} \end{bmatrix} \quad \boldsymbol{R} = \begin{bmatrix} 0 & c_{13} & e_{31} \\ c_{44} & 0 & 0 \\ e_{15} & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{W} = \begin{bmatrix} c_{44} & 0 & 0 \\ 0 & c_{33} & e_{33} \\ 0 & e_{33} & -\varepsilon_{33} \end{bmatrix}$$

and $\boldsymbol{\beta}_1 = (\beta_{11},\ 0,\ 0)^{\mathrm{T}}$, $\boldsymbol{\beta}_2 = (0,\ \beta_{33},\ \tau_3)^{\mathrm{T}}$.

Based on the given constants, we have

$$\mu_1 = 1.8267\mathrm{i} \quad \mu_2 = 0.8303\mathrm{i}$$

$$\mu_3 = 0.5941\mathrm{i} \quad \mu_t = 0.816\mathrm{i}$$

and

$$\boldsymbol{A}=\begin{bmatrix}3.1992\times10^{-6}(1-\mathrm{i}) & 1.5953\times10^{-6}(-1+\mathrm{i}) & 1.7736\times10^{-6}(-1-\mathrm{i})\\ 1.0843\times10^{-6}(1+\mathrm{i}) & 2.0528\times10^{-6}(-1-\mathrm{i}) & 3.8023\times10^{-6}(1-\mathrm{i})\\ 1.4388\times10^{4}(1+\mathrm{i}) & 5.7446\times10^{4}(1+\mathrm{i}) & 2.5718\times10^{4}(-1+\mathrm{i})\end{bmatrix}$$

$$\boldsymbol{B}=\begin{bmatrix}8.9466\times10^{4}(1+\mathrm{i}) & 5.2508\times10^{4}(-1-\mathrm{i}) & 6.7649\times10^{4}(1-\mathrm{i})\\ 4.9878\times10^{4}(-1+\mathrm{i}) & 6.3241\times10^{4}(1-\mathrm{i}) & 1.1386\times10^{5}(1+\mathrm{i})\\ 1.1740\times10^{-6}(1-\mathrm{i}) & 5.1536\times10^{-6}(1-\mathrm{i}) & 2.4475\times10^{-6}(1+\mathrm{i})\end{bmatrix}$$

Fig. 2 shows the normalized hoop stress on the rim of hole $\lambda_{22}\sigma_{\varphi\varphi}/10\beta_{11}$ versus orientation θ with different value of λ_1 at $q_1^{\infty}=1\ \mathrm{W\cdot m^{-2}}$, $r=10^{-4}$ m, $\lambda_2=0$. It is seen that $\lambda_{22}\sigma_{\varphi\varphi}/10\beta_{11}$ reaches its maximum when $\theta=0$ and π. The value of $\lambda_{22}\sigma_{\varphi\varphi}/10\beta_{11}$ equals to zero when $\theta=\pi/2$ or $\theta=3\pi/2$. The hoop stress $\lambda_{22}\sigma_{\varphi\varphi}/10\beta_{11}$ decreases with the increasement of the thermal conduction coefficient, which means if the heat flow may pass through the hole easily, the hoop stress around the hole will be low. On the contrary, the gathering heat flow enhances the hoop stress.

Fig. 3 shows the normalized hoop stress on the rim of hole $\lambda_{22}\sigma_{\varphi\varphi}/10\beta_{11}$ versus orientation θ with different value of λ_2 at $q_2^{\infty}=1\ \mathrm{W\cdot m^{-2}}$, $r=10^{-4}$ m, $\lambda_1=0$. It is seen that the value of $\lambda_{22}\sigma_{\varphi\varphi}/10\beta_{11}$ equals to zero when $\theta=0$ and $\theta=\pi$. At $\theta=1$, 2.4, 4.14 and 5.54, $|\sigma_{\varphi\varphi}|$ reaches its maximum. When $\theta=\pi/2$ and $\theta=3\pi/2$, $|\sigma_{\varphi\varphi}|$ reaches its second largest value.

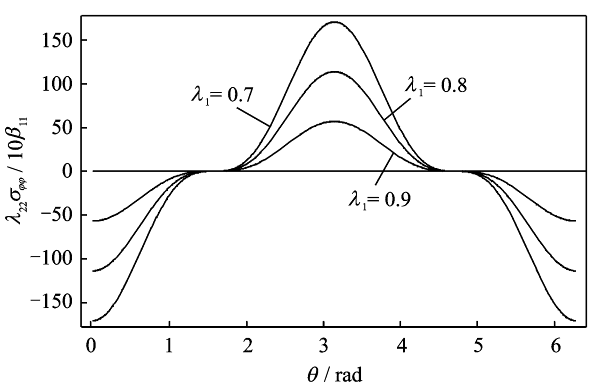

Fig. 2 Curves for normalized hoop stress $\lambda_{22}\sigma_{\varphi\varphi}/10\beta_{11}$ versus orientation θ at $q_1^{\infty}=1\ \mathrm{W\cdot m^{-2}}$, $r=10^{-4}$ m, $\lambda_2=0$

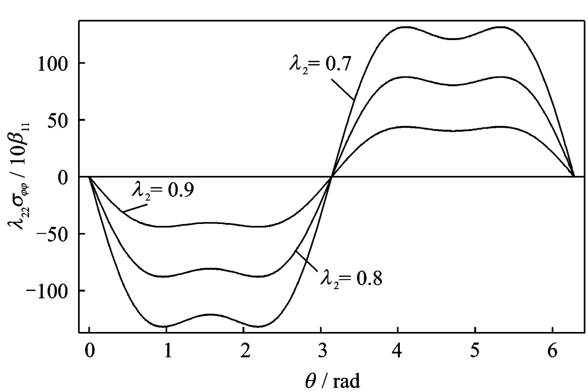

Fig. 3 Curves for normalized hoop stress $\lambda_{22}\sigma_{\varphi\varphi}/10\beta_{11}$ versus orientation θ at $q_2^{\infty}=1\ \mathrm{W\cdot m^{-2}}$, $r=10^{-4}$ m, $\lambda_1=0$

6 Heat Flow on Hole Rim

From Eqs. (20—21) we have

$$q_1=2\mathrm{Re}[\mathrm{i}\mu_t\kappa_t g''(\sigma)],\ q_2=2\mathrm{Re}[\mathrm{i}\kappa_t g''(\sigma)] \tag{58}$$

$$q_\tau=q_2\cos\theta-q_1\sin\theta,\ q_n=q_2\sin\theta+q_1\cos\theta \tag{59}$$

Fig. 4 shows the normal component of heat flow q_n versus orientation θ with different value of λ_2 at $q_1^{\infty}=q_2^{\infty}=1\ \mathrm{W\cdot m^{-2}}$, $r=10^{-4}$ m, $\lambda_1=1$. It is seen that λ_2 has a significant influence on q_n. The orientation θ increases with the increasement of λ_2 when q_n reaches its maximum, but when $\theta=0$ or $\theta=\pi$, q_n and λ_2 are independent.

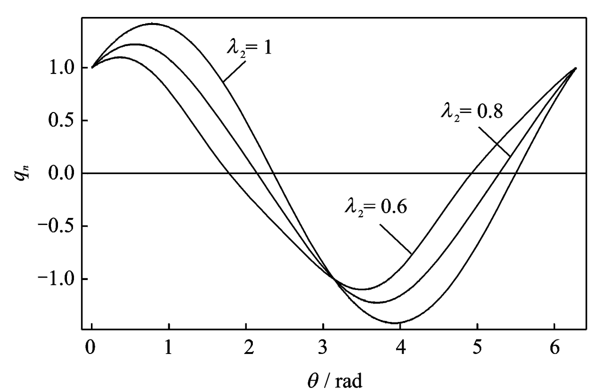

Fig. 4 Curves for heat flow q_n versus orientation θ with different value of λ_2 at $q_1^{\infty}=q_2^{\infty}=1\ \mathrm{W\cdot m^{-2}}$, $r=10^{-4}$ m, $\lambda_1=1$

The curves for variations of heat flow q_n versus orientation θ with different value of λ_1 are shown in Fig. 5 for the case of $q_1^{\infty}=q_2^{\infty}=1\ \mathrm{W\cdot m^{-2}}$, $r=10^{-4}$ m and $\lambda_2=1$. The orientation θ decreases with the increasement of λ_1 when q_n reaches its maximum, but when $\theta=\pi/2$ or $\theta=3\pi/2$,

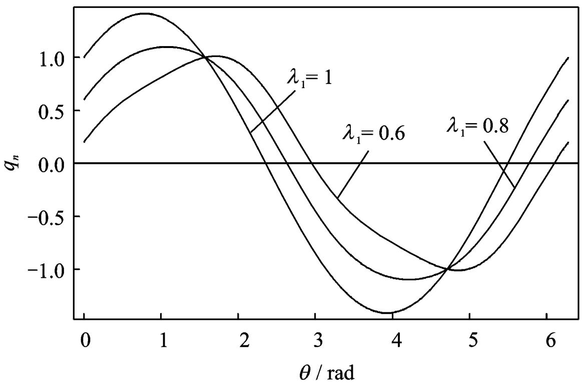

Fig. 5 Curves for heat flow q_n versus orientation θ with different value of λ_1 at $q_1^\infty = q_2^\infty = 1$ W · m^{-2}, $r = 10^{-4}$ m, $\lambda_2 = 1$

q_n and λ_1 are independent.

The curves for variations of heat flow q_τ versus orientation θ with different value of λ_2 are shown in Fig. 6 for the case of $q_1^\infty = q_2^\infty = 1$ W · m^{-2} and $r = 10^{-4}$ m and $\lambda_1 = 1$. It is seen that λ_2 has a significant influence on q_τ. The orientation θ decreases with the increasement of λ_2 when q_τ reaches its maximum, but when $\theta = \pi/2$ or $\theta = 3\pi/2$, q_τ and λ_2 are independent.

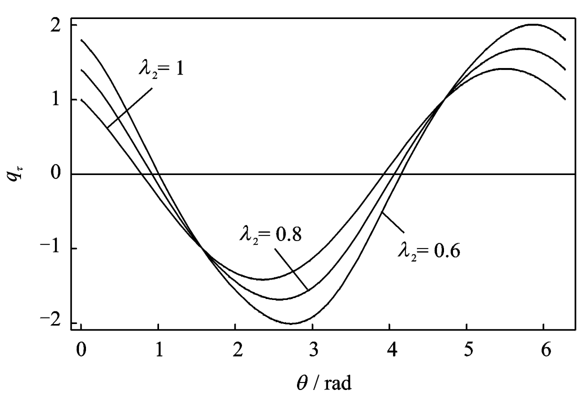

Fig. 6 Curves for heat flow q_τ versus orientation θ with different value of λ_2 at $q_1^\infty = q_2^\infty = 1$ W · m^{-2}, $r = 10^{-4}$ m, $\lambda_1 = 1$

Fig. 7 shows the tangential component of heat flow q_τ versus orientation θ with different value of λ_1 at $q_1^\infty = q_2^\infty = 1$ W · m^{-2}, $r = 10^{-4}$ m and $\lambda_2 = 1$. It is seen that $|q_\tau|$ decreases with the increasement of λ_1. And the orientation θ increases with the increasement of λ_1 when q_τ reaches its maximum, but when $\theta = 0$ or $\theta = \pi$, q_τ and λ_1 are independent.

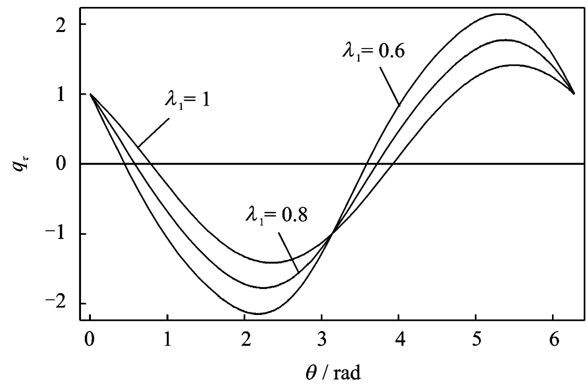

Fig. 7 Curves for heat flow q_τ versus orientation θ with different value of λ_1 at $q_1^\infty = q_2^\infty = 1$ W · m^{-2}, $r = 10^{-4}$ m, $\lambda_2 = 1$

7 Conclusions

(1) The thermal boundary condition has significant effect influence on the hoop stress and heat flow around a hole in thermopiezoelectric materials under a thermal loading.

(2) The hoop stress decreases dramatically with the increasement of the thermal conduction coefficient, which means if the heat flow may pass through the hole easily, the hoop stress around the hole will be low. On the contrary, the gathering heat flow enhances the hoop stress.

(3) The orientation θ when q_n (or q_τ) reaches its maximum changes with the variation of the thermal conduction coefficient λ_1 (and λ_2). But at certain points, q_n (or q_τ) and λ_1 (and λ_2) are independent.

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