Potential Method in the Theory of Thermoelasticity with Microtemperatures for Microstretch Solids

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Abstract: The linear equilibrium theory of thermoelasticity with microtemperatures for microstretch solids is considered. The basic internal and external boundary value problems (BVPs) are formulated and uniqueness theorems are given. The single-layer and double-layer thermoelastic potentials are constructed and their basic properties are established. The integral representation of general solutions is obtained. The existence of regular solutions of the BVPs is proved by means of the potential method (boundary integral method) and the theory of singular integral equations.

Key words: thermoelasticity with microtemperatures; existence and uniqueness theorems; microstretch solids

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1 Introduction

In the last decades several continuum theories with microstructures have been formulated[1,2]. The linear theory of thermoelasticity with microtemperatures for materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures was presented in Ref. [3]. In the recent years this theory has been intensively studied. The fundamental solutions in the theory of thermoelasticity with microtemperatures were constructed[4]. The representations of Galerkin type and general solutions of equations of motion and steady vibrations in this theory were obtained[5]. The exponential stability of solution of equations of the theory of thermoelasticity with microtemperatures was constructed[6]. The basic theorems in the linear equilibrium theory of thermoelasticity with microtemperatures were proved[7]. The basic three-dimensional (3D) internal and external BVPs of steady vibrations were investigated using the potential method and the theory of singular integral equations[8-10]. The boundary value problems (BVPs) in the dynamical theory of thermoelasticity with microtemperatures were studied[11]. The behavior of shock waves in a thermoelastic body with inner structure and microtemperatures was established[12]. Singular surfaces and plane harmonic waves in thermoelastic solid with microtemperatures were studied in Refs. [13-15].

The uniqueness and existence theorems for the 2D BVPs in the theory of thermoelasticity with microtemperatures were proved[16]. The hirarchical models for elastic prismatic shells with microtemperatures were considered[17].

The theory thermoelasticity with microtemperatures for microstretch elastic solids was formulated in Ref. [18]. The fundamental solutions of equations of this theory were constructed[19]. The uniqueness theorems in the equilibrium theory of thermoelasticity with microstructures for microstretch solids were proved in Ref. [20]. Recently, the representation of Galerkin type solution for the system of equations of steady vibrations in the linear theory of thermoelasticity with microstructures for microstretch solids has been obtained[21].

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The theory of micropolar thermoelasticity with microtemperatures was presented in Ref. [22]. The existence and asymptotic behavior of the solutions in this theory were proved[23]. The linear theory of thermoelastic bodies with microstructure and microtemperatures which permits the transmission of heat as thermal waves at finite speed was constructed[24]. The uniqueness theorems in the theory of thermoelasticity of porous media with microtemperatures are presented[25]. An extensive review and the basic results in the microcontinuum field theories are given in Refs. [1,2].

In this paper, authors consider the linear equilibrium theory of thermoelasticity with microtemperatures for microstretch solids[18]. The basic internal and external BVPs are formulated and uniqueness theorems are given. The single-layer and double-layer thermoelastic potentials are constructed and their basic properties are established. The integral representation of general solutions is obtained. The existence of regular solutions of the BVPs by means of the potential method and the theory of singular integral equations are proved.

2 Basic Equations

Let $x=(x_1,x_2,x_3)$ be the point of the Euclidean 3D space. An isotropic and homogeneous microstretch elastic solid is considered. The system of homogenous equations of the linear equilibrium theory of thermoelasticity with microtemperatures for such material can be written as[18]

$$\begin{aligned}&\mu\Delta\boldsymbol{u}+(\lambda+\mu)\mathrm{graddiv}\boldsymbol{u}-\beta\mathrm{grad}\theta+b\mathrm{grad}\varphi=0\\&k_6\Delta\boldsymbol{w}+(k_4+k_5)\mathrm{graddiv}\boldsymbol{w}-k_3\mathrm{grad}\theta-k_2\boldsymbol{w}=0\\&k\Delta\theta+k_1\mathrm{div}\boldsymbol{w}=0\\&\gamma\Delta\varphi-b\mathrm{div}\boldsymbol{u}-d\mathrm{div}\boldsymbol{w}+m\theta-\xi\varphi=0\end{aligned}\qquad(1)$$

where $\boldsymbol{u}=(u_1,u_2,u_3)$ is the displacement vector, $\boldsymbol{w}=(w_1,w_2,w_3)$ the microtemperature vector; θ the temperature measured from the constant absolute temperature T_0; φ the microdilatation function. $\lambda,\mu,\beta,\gamma,\xi,b,d,m,k,k_1,k_2,\cdots,k_6$ are the constitutive coefficients and Δ is the Laplacian operator.

It is assumed that the constitutive coefficients satisfy the following conditions[18]

$$\begin{aligned}&\mu>0,\ 3\lambda+2\mu>0,\ \gamma>0,\ k>0\\&k_6\pm k_5>0,\ 3k_4+k_5+k_6>0\\&(3\lambda+2\mu)\xi>3b^2,\ (k_1+T_0k_3)^2<4T_0kk_1\end{aligned}$$

3 BVP Uniqueness Theorems

In this section, the internal and external BVPs of the linear equilibrium theory of thermoelasticity with microtemperatures for isotropic and homogeneous microstretch solids are formulated.

Let S be a closed surface surrounding the finite domain Ω^+ in E^3, $\bar{\Omega}^+=\Omega^+\cup S$, $\Omega^-=E^3\setminus\bar{\Omega}^+$, $\bar{\Omega}^-=\Omega^-\cup S, S\in C^{2,\nu}$, $0<\nu\leqslant1$.

A vector function $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\varphi,\theta)=(U_1,U_2,\cdots,U_8)$ is called regular in Ω^+, if $U_j\in C^2(\Omega^+)\cap C^1(\bar{\Omega}^+)$, and $\boldsymbol{U}$ is called regular in Ω^-, if $U_j\in C^2(\Omega^-)\cap C^1(\bar{\Omega}^-)$, $U_j(\boldsymbol{x})=O(|\boldsymbol{x}|^{-1})$, $\frac{\partial}{\partial x_l}U_j(\boldsymbol{x})=O(|\boldsymbol{x}|^{-1})$, $j=1,2,\cdots,8$; $l=1,2,3$ for $|\boldsymbol{x}|\gg1$.

The basic internal and external BVPs of the linear equilibrium theory of thermoelasticity with microtemperatures for microstretch solid are formulated as follows.

Problem $(I)_f^+$: Find a regular solution to system (1) in Ω^+ that satisfies the boundary condition

$$\lim_{\Omega^+\ni\boldsymbol{x}\to z\in S}\boldsymbol{U}(\boldsymbol{x})\equiv\{\boldsymbol{U}(z)\}^+=\boldsymbol{f}(z)\qquad(2)$$

Problem $(II)_f^+$: Find a regular solution to system (1) in Ω^+ that satisfies the boundary condition

$$\lim_{\Omega^+\ni\boldsymbol{x}\to z\in S}\boldsymbol{PU}(\boldsymbol{x})\equiv\{\boldsymbol{PU}(z)\}^+=\boldsymbol{f}(z)$$

Problem $(I)_f^-$: Find a regular solution to system (1) in Ω^+ that satisfies the boundary condition

$$\lim_{\Omega^-\ni\boldsymbol{x}\to z\in S}\boldsymbol{U}(\boldsymbol{x})\equiv\{\boldsymbol{U}(z)\}^-=\boldsymbol{f}(z)$$

Problem $(II)_f^-$: Find a regular solution to system (1) in Ω^- that satisfies the boundary condition

$$\lim_{\Omega^-\ni\boldsymbol{x}\to z\in S}\boldsymbol{PU}(\boldsymbol{x})\equiv\{\boldsymbol{PU}(z)\}^-=\boldsymbol{f}(z).$$

where $\boldsymbol{P}$ is the stress operator in the linear theory of thermoelasticity with microtemperatures for microstretch solid[18] and $\boldsymbol{f}$ the eight-component known vector function on S.

We have the following results[20].

Theorem 1 The internal BVP $(I)_f^+$ admits at most one regular solution.

Theorem 2 Any two regular solutions of the internal BVP $(II)_f^+$ differ only by an additive vector $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\varphi,\theta)$ where
$\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{a}'+[\boldsymbol{a}''\times\boldsymbol{x}]+d_1\boldsymbol{x}$, $\boldsymbol{w}(\boldsymbol{x})=0$,
$\varphi(\boldsymbol{x})=d_2$, $\theta(\boldsymbol{x})=c_1$
for $\boldsymbol{x}\in\Omega^+$, $\boldsymbol{a}'$ and $\boldsymbol{a}''$ being arbitrary real constant three-component vectors, c_1 an arbitrary real constant

$$d_1=\frac{\beta\xi-bm}{(3\lambda+2\mu)\xi-3b^2}c_1,\ d_2=\frac{(3\lambda+2\mu)m-3b\beta}{(3\lambda+2\mu)\xi-3b^2}c_1$$

Theorem 3 The external BVP $(K)_f^+$ admits at most one regular solution, where $K=I,II$.

4 Thermoelastic Potentials and Singular Integral Operator

In this section authors present the basic properties of the thermoelastic potentials. In the sequel the following notations are used

$$(1)\ Z^{(1)}(x,\boldsymbol{g})=\int_S\boldsymbol{\Gamma}(x-y)\boldsymbol{g}(y)\mathrm{d}_yS \qquad (3)$$

$Z^{(1)}$ is a single-layer potential;

$$(2)\ Z^{(2)}(x,\boldsymbol{g})=\int_S[\widetilde{\boldsymbol{P}}\boldsymbol{\Gamma}^*(x-y)]^*\boldsymbol{g}(y)\mathrm{d}_yS \qquad (4)$$

$Z^{(2)}$ is a double-layer potential, where the matrix $\boldsymbol{\Gamma}=(\Gamma_{lj})_{8\times8}$ is the fundamental solution of system (1), $\widetilde{\boldsymbol{P}}=(\widetilde{P}_{lj})_{8\times8}$ is the first-order matrix differential operator, and $\boldsymbol{g}$ is the eight-component vector function.

In Ref. [19], the matrix $\boldsymbol{\Gamma}$ is constructed in terms of elementary functions and basic properties are established.

On the basis of Eqs. (3,4) it can obtain the following formulae of integral representations of regular vector (Somigliana type formulae) in the equilibrium theory of thermoelasticity with microtemperatures for microstretch solid

$$\boldsymbol{U}(x)=\boldsymbol{Z}^{(2)}(x,\{\boldsymbol{U}\}^+)-\boldsymbol{Z}^{(1)}(x,\{\widetilde{\boldsymbol{P}}\boldsymbol{U}\}^+)$$

for $x\in\Omega^+$, and

$$\boldsymbol{U}(x)=-\boldsymbol{Z}^{(2)}(x,\{\boldsymbol{U}\}^-)+\boldsymbol{Z}^{(1)}(x,\{\widetilde{\boldsymbol{P}}\boldsymbol{U}\}^-)$$

for $x\in\Omega^-$.

We have the following results:

Theorem 4 If $\boldsymbol{g}\in C^{1,\nu_1}(S)$, $0<\nu_1<\nu\leqslant1$, then

(a) $\boldsymbol{Z}^{(1)}(\cdot,\boldsymbol{g})\in C^{0,\nu_1}(E^3)\cap C^{2,\nu_1}(\bar{\Omega}^\pm)\cap C^\infty(\Omega^\pm)$;

(b) $\boldsymbol{Z}^{(1)}(x,\boldsymbol{g})$ is a solution of the system (1);

(c) $\boldsymbol{PZ}^{(1)}(z,\boldsymbol{g})$ is a singular integral for $z\in S$;

(d) $\{\boldsymbol{PZ}^{(1)}(z,\boldsymbol{g})\}^\pm=\mp\frac{1}{2}\boldsymbol{g}(z)+\boldsymbol{PZ}^{(1)}(z,\boldsymbol{g})$.

Theorem 5 If $\boldsymbol{g}\in C^{1,\nu_1}(S)$, $0<\nu_1<\nu\leqslant1$, then

(a) $\boldsymbol{Z}^{(2)}(\cdot,\boldsymbol{g})\in C^{1,\nu_1}(\bar{\Omega}^\pm)\cap C^\infty(\Omega^\pm)$;

(b) $\boldsymbol{Z}^{(2)}(x,\boldsymbol{g})$ is a solution of the system (1);

(c) $\boldsymbol{Z}^{(2)}(z,\boldsymbol{g})$ is a singular integral for $z\in S$;

(d) $\{\boldsymbol{Z}^{(2)}(z,\boldsymbol{g})\}^\pm=\mp\frac{1}{2}\boldsymbol{g}(z)+\boldsymbol{Z}^{(2)}(z,\boldsymbol{g})$.

Theorems 4 and 5 can be proved similarly to the corresponding theorems in the classical theory of thermoelasticity (for details see Ref. [26], Chapter X).

Here introduce the notation

$$M\boldsymbol{g}(z)=\frac{1}{2}\boldsymbol{g}(z)+\boldsymbol{Z}^{(2)}(z,\boldsymbol{g})\quad\text{for}\quad z\in S$$

Theorem 6 The singular integral operator M is of the normal type with an index equal to zero, i. e. Fredholm's theorems are valid for the following singular integral equation

$$M\boldsymbol{g}(z)=\boldsymbol{f}(z)\quad\text{for } z\in S \qquad (5)$$

For the definitions of a normal type singular integral operator, the symbol and the index of operators see, e. g., Ref. [26], Chapter IV.

5 Existence Theorem

The existence theorem of regular solution of the problems $(I)_f^+$ is proved by means of the potential method and the theory of singular integral equations.

In this paper, authors seek a regular solution of Problems (1—3) in the form of double-layer potential

$$\boldsymbol{U}(x)=\boldsymbol{Z}^{(2)}(x,\boldsymbol{g})\ \text{for } x\in\Omega^+ \qquad (6)$$

By virtue of Theorem 5 and the boundary condition Eq. (2), from Eq. (6) singular integral equation (5) is obtained. On the basis of Theorem 6, Fredholm′s theorems are valid for the integral equation (5).

In addition, the homogeneous adjoint singular integral equation of Eq. (5) has the following form

$$\frac{1}{2}\boldsymbol{h}(z)+\boldsymbol{PZ}^{(1)}(z,\boldsymbol{h})=0 \text{ for } z\in S \qquad (7)$$

where $\boldsymbol{h}$ is the eight-component vector function on S. Eq. (7) has only a trivial solution $\boldsymbol{h}(z)\equiv 0$. Hence, the integral equation (5) is always solvable for an arbitrary vector $\boldsymbol{f}$ and the following theorem is proved.

Theorem 7 A regular solution of the Problem $(I)_f^+$ exists, which is unique and can be represented by double-layer potential Eq. (6). Here in Eq. (6) $\boldsymbol{g}$ is a solution of the singular integral equation (5) which is always solvable for an arbitrary vector $\boldsymbol{f}$.

The existence theorems for the BVPs $(II)_f^+$, $(I)_f^-$, and $(II)_f^-$ are proved similarly.

6 Conclusions

(1) By the method developed in this paper, it is possible to investigate the steady vibrations and dynamical BVPs in the linear theory of thermoelasticity with microtemperatures for microstretch solids.

(2) By the above mentioned method, it is possible to prove the existence and uniqueness theorems in the modern linear theories of elasticity and thermoelasticity for materials with microstructure.

(3) On the basis of results of this paper, it is possible to obtain numerical solutions of the BVPs of the linear theory in thermoelasticity with microtemperatures for microstretch solids by using of the boundary element method.

References:

[1] Eringen A C. Microcontinuum field theories I: Foundation and solids[M]. New York, Berlin, Heidelberg: Springer-Verlag, 1999.

[2] Iesan D. Thermoelastic models of continua[M]. Boston, Dordrecht, London, Kluwer: Academic Publishers, 2005.

[3] Iesan D, Quintanilla R. On the theory of thermoelasticity with microtemperatures[J]. J Thermal Stresses, 2000,23:199-215.

[4] Svanadze M. Fundamental solutions of the equations of the theory of thermoelasticity with microtemperatures[J]. J Thermal Stresses, 2004,27:151-170.

[5] Scalia A, Svanadze M. On the representation of solutions of the theory of thermoelasticity with microtemperatures[J]. J Thermal Stresses, 2006, 29: 849-864.

[6] Casas P S, Quintanilla R. Exponential stability in thermoelasticity with microtemperatures[J]. Int J Engng Sci, 2005,43:33-47.

[7] Scalia A, Svanadze M, Tracinà R. Basic theorems in the equilibrium theory of thermoelasticity with microtemperatures[J]. J Thermal Stresses, 2010,33:721-753.

[8] Svanadze M. Boundary value problems of the theory of thermoelasticity with microtemperatures[J]. Proc Appl Math Mech, 2003,3:188-189.

[9] Scalia A, Svanadze M. Potential method in the linear theory of thermoelasticity with microtemperatures [J]. J Thermal Stresses, 2009,32:1024-1042.

[10] Svanadze M. Boundary value problems of steady vibrations in the theory of thermoelasticity with microtemperatures[J]. Proc Appl Math Mech, 2011,11: 443-444.

[11] Svanadze M. On the linear theory of thermoelasticity with microtemperatures[J]. Tech Mech, 2012, 32: 564-576.

[12] Svanadze M. Boundary integral method in the dynamical theory of thermoelasticity with microtemperatures[C]//Troch I, Breitenecker F (Eds.) 7th Vienna Int Confer Math Mod, 2012. [S. l.]: Vienna Univ Tech, ARGESIM Rep. No. AR-S38.

[13] Iesan D. Singular surfaces in the theory of thermoelasticity with microtemperatures[J]. J Thermal Stresses, 2009,32:1279-1292.

[14] Svanadze M. Plane waves in the theory of thermoelasticity with microtemperatures[C]//9th International Congress on Thermal Stresses. Budapest, Hungary: [s. n.], 2011.

[15] Steeb H, Singh J, Tomar S K. Time harmonic waves in thermoelastic material with microtemperatures[J]. Mech Res Commun, 2013,48:8-18.

[16] Bitsadze L, Jaiani, G. Some basic boundary value

problems of the plane thermoelasticity with microtemperatures[J]. Math Meth Appl Sci, 2013, DOI: 10.1002/mma.2652.

[17] Jaiani G. Zero order approximation of hirarchical models for elastic prismatic shells with microtemperatures[J]. Bulletin of TICMI, 2012, 16: 66-73.

[18] Iesan D. On a theory of micromorphic elastic solids with microtemperatures[J]. J Thermal Stresses, 2001, 24: 737-752.

[19] Svanadze M. Fundamental solutions in the theory of micromorphic elastic solids with microtemperatures [J]. J Thermal Stresses, 2004, 27: 345-366.

[20] Scalia A, Svanadze M. Uniqueness theorems in the equilibrium theory of thermoelasticity with microstructures for microstretch solids[J]. J Mech Mater Struct, 2011, 6: 1295-1311.

[21] Svanadze M, Tracinà R. Representations of solutions in the theory of thermoelasticity with microtemperatures for microstretch solids[J]. J Thermal Stresses, 2011, 34: 161-178.

[22] Iesan D. Thermoelasticity of bodies with microstructure and microtemperatures[J]. 2007, 44: 8648-8662.

[23] Aouadi M. Some theorems in the isotropic theory of microstretch thermoelasticity with microtemperatures [J]. J Thermal Stresses, 2008, 31: 649-662.

[24] Iesan D, Quintanilla R. On thermoelastic bodies with inner structure and microtemperatures[J]. J Math Anal Appl, 2009, 354: 12-23.

[25] Quintanilla R. Uniqueness in thermoelasticity of porous media with microtemperatures[J]. Arch Mech, 2009, 61: 371-382.

[26] Kupradze V D, Gegelia T G, Basheleishvili M O, et al. Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity[M]. Amsterdam, New York, Oxford: North-Holland Publ Com, 1979.

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