Thermal Stresses and Theorem on Decomposition

Valeriy Lokhov * , *Yuriy Nyashin*

Department of Theoretical Mechanics and Biomechanics,

Perm National Research Polytechnic University, Perm, Russia

(Received 15 October 2013; revised 10 February 2014; accepted 20 February 2014)

Abstract: The thermal expansion strain is considered as a special case of eigenstrain. The authors proved the theorem on decomposition of eigenstrain existing in a body into two constituents: Impotent eigenstrain (not causing stress in any point of a body) and nilpotent eigenstrain (not causing strain in any point of a body). According to this theorem, the thermal stress can be easily found through the nilpotent eigenstrain. If the eigenstrain is an impotent one, the thermal stress vanishes. In this case, the eigenstrain must be compatible. The authors suggest a new approach to measure of eigenstrain incompatibility and hence to estimate of thermal stresses.

Key words: eigenstrain; thermal stresses; decomposition; impotent eigenstrain; nilpotent eigenstrain; functional space

CLC number: O34 **Document code**: A **Article ID**: 1005-1120(2014)02-0185-05

1 Introduction

According to Reissner, 1931[1], any tensor of geometrically linear strain existing in a body can be presented as a sum of elastic strain (which can be found according to Hooke's law) and tensor of eigenstrain. Examples of eigenstrain are: Thermal expansion strain, plastic strain, creep strain, strain due to phase transformations, and growth strain in living tissues, etc.

Therefore, we have

$$\tilde{\varepsilon}=\tilde{\varepsilon}^{e}+\tilde{\varepsilon}^{*}, \qquad \tilde{\varepsilon}(\boldsymbol{u})=\frac{1}{2}(\nabla \boldsymbol{u}+\boldsymbol{u} \nabla)$$

$$\tilde{\varepsilon}^{e}=\tilde{\tilde{\boldsymbol{C}}}^{-1} \cdot\cdot\, \tilde{\sigma} \tag{1}$$

where total strain tensor $\tilde{\varepsilon}$ is related to the displacement $\boldsymbol{u}$ and elastic strain tensor $\tilde{\varepsilon}^{e}$ is linearly related to the stress tensor $\tilde{\sigma}$ by Hooke's law with $\tilde{\tilde{\boldsymbol{C}}}^{-1}$ denoting the elastic compliance tensor.

2 Generalized Formulation of Basic Boundary Value Problem in Linearized Elasticity with Eigenstrain

Stress is given by the generalized Hooke's law (eliminating elastic strain in Eq. (1))

$$\tilde{\sigma}=\tilde{\tilde{\boldsymbol{C}}} \cdot\cdot\, (\tilde{\varepsilon}(\boldsymbol{u})-\tilde{\varepsilon}^{*}) \tag{2}$$

where $\boldsymbol{u}\in (W_2^1(\boldsymbol{\Omega}))^3$, $\boldsymbol{u}=0$, $\boldsymbol{x}\in \Gamma_u$, and the work of internal and external forces must vanish, namely

$$\int_{\Omega}\tilde{\sigma} \cdot\cdot\, \tilde{\varepsilon}(\boldsymbol{w})\mathrm{d}V-\int_{\Gamma_\sigma}\boldsymbol{t}\cdot \boldsymbol{w}\mathrm{d}S-\int_{\Omega}\boldsymbol{b}\cdot \boldsymbol{w}\mathrm{d}V=0$$

$$\forall\, \boldsymbol{w} \in (W_2^1(\Omega))^3, \quad \boldsymbol{w}=0,\ \boldsymbol{x}\in \Gamma_u \tag{3}$$

where $\boldsymbol{\Omega}$ is a bounded region with boundary (assumed to be sufficiently smooth) Γ. The boundary Γ is divided into disjoint parts: $\Gamma=\Gamma_u\cup\Gamma_\sigma$. Kinematical boundary conditions are set at the part Γ_u of the boundary and the traction vector $\boldsymbol{t}$ is prescribed at the part Γ_σ

$$\boldsymbol{u}=0,\ \boldsymbol{x}\in \Gamma_u \tag{4}$$

$$\boldsymbol{u}\cdot \tilde{\sigma}=\boldsymbol{t},\ \boldsymbol{x}\in \Gamma_\sigma \tag{5}$$

Further, W_2^1 is the Sobolev space of functions with generalized derivatives, and the functions and their derivatives are squared summable. Strains $\tilde{\varepsilon}(\boldsymbol{u})$ and $\tilde{\varepsilon}(\boldsymbol{v})$ are defined by linearized geometrical relations where generalized derivatives are understood. Values of displacements $\boldsymbol{u}$ and $\boldsymbol{v}$ at the boundary are calculated by means of the trace operator. Also, we suppose: $\boldsymbol{t}\in (L_2(\Gamma_\sigma))^3$, $\boldsymbol{b}\in (L_2(\Omega))^3$, $\tilde{\varepsilon}\in (L_2(\Omega))^6$, $\boldsymbol{C}_{ijkl}(i,j,k,l=1,2,3)$ are piecewise continuous

* **Corresponding author**: Valeriy Lokhov, Associate Professor, E-mail: Valeriy.Lokhov@yandex.ru.

functions of coordinates.

The generalized solution has an analogous sense with principle of virtual displacement and does not contain the derivatives of stress and strain with respect to coordinates. It can be easily shown that classical solution is a generalized one, and in the case of adequate smoothness of the stress $\tilde{\sigma}$, the generalized solution is a classical one.

Existence and uniqueness of the rewritten generalized solution of the problem have been shown by Duvant and Lions[2].

3 Impotent and Nilpotent Eigenstrains

The fundamental definitions of impotent and nilpotent eigenstrains are given in this section.

Impotent eigenstrain $\tilde{\varepsilon}_u^*$ [3] does not cause stress in any point of the body. Consequently, impotent eigenstrain is equal to total strain, namely

$$\tilde{\varepsilon}=\tilde{\varepsilon}_u^* \tag{6}$$

Nilpotent eigenstrain $\tilde{\varepsilon}_\sigma^*$ [4] does not cause deformation in any point of the body, i. e.

$$\tilde{\varepsilon}=\tilde{\varepsilon}^e+\tilde{\varepsilon}_\sigma^*=\tilde{\tilde{\boldsymbol{C}}}^{-1}\cdot\cdot\,\tilde{\sigma}+\tilde{\varepsilon}_\sigma^*=0 \tag{7}$$

$$\tilde{\varepsilon}_\sigma^*=-\tilde{\tilde{\boldsymbol{C}}}^{-1}\cdot\cdot\,\tilde{\sigma} \tag{8}$$

In the absence of both volume force $\boldsymbol{b}$ and surface traction $\boldsymbol{t}$, the classical formulation of the problems for impotent and nilpotent eigenstrains have forms:

(1) For impotent eigenstrain $\tilde{\varepsilon}_1^*$

$$\begin{aligned}&\tilde{\sigma}_1=0,\ \boldsymbol{x}\in\Omega=\Omega\cup\Gamma\\&\tilde{\varepsilon}_1=\tilde{\varepsilon}_1^*,\ \boldsymbol{x}\in\Omega\\&\tilde{\varepsilon}_1=\frac{1}{2}(\nabla\boldsymbol{u}_1+\boldsymbol{u}_1\nabla),\ \boldsymbol{x}\in\Omega\\&\boldsymbol{u}_1=0,\ \boldsymbol{x}\in\Gamma_u\end{aligned} \tag{9}$$

(2) For nilpotent eigenstrain $\tilde{\varepsilon}_2^*$

$$\begin{aligned}&\nabla\cdot\tilde{\sigma}_2=0,\ \boldsymbol{x}\in\Omega\\&\tilde{\sigma}_2=-\tilde{\tilde{\boldsymbol{C}}}\cdot\cdot\,\tilde{\varepsilon}_2^*,\ \boldsymbol{x}\in\Omega\\&\boldsymbol{n}\cdot\tilde{\sigma}_2=0,\ \boldsymbol{x}\in\Gamma_\sigma\\&\boldsymbol{u}_2=0,\ \tilde{\varepsilon}_2=0,\ \boldsymbol{x}\in\Omega\end{aligned} \tag{10}$$

4 Function Space of Eigenstrain

Let us consider the set H of symmetric tensors of the second rank. The components of the tensors are assumed to be real functions of the spatial coordinates and belong to the function space L_2.

The scalar product satisfying symmetry, linearity and positivity in space H is defined by

$$(\tilde{A},\tilde{B})=\int_\Omega A_{ij}\boldsymbol{C}_{ijkl}B_{kl}\,\mathrm{d}V \tag{11}$$

The norm in space H is defined by means of the scalar product

$$\|\tilde{A}\|=\sqrt{(\tilde{A},\tilde{A})} \tag{12}$$

This space is a Hilbert space, or energy space[5].

Subspace H_u is introduced by the following definition that a symmetric tensor $\tilde{f}\in H$ belongs to subspace H_u if there exists such vector-function $\exists\,\boldsymbol{u}\in(W_2^1(\Omega))^3$, $\boldsymbol{u}=0$, $\boldsymbol{x}\in\Gamma_u$, and

$$\tilde{f}=\frac{1}{2}(\nabla\boldsymbol{u}+\boldsymbol{u}\nabla) \tag{13}$$

The physical sense of subspace H_u lies in the fact that this subspace is a set of compatible tensors of strain when the corresponding displacement $\boldsymbol{u}$ is equal to zero at the immovable supports.

It can be proved that condition $\tilde{\varepsilon}^*\in H_u$ is a necessary and sufficient one for $\tilde{\sigma}=0$ in every point of a body due to action of eigenstrain. In particular, it is true for thermal strain and stress.

Further, we will prove this assertion. If $\tilde{\sigma}=0$, then from Eq. (2) it is follows that $\tilde{\varepsilon}(\boldsymbol{u})-\tilde{\varepsilon}^*=\tilde{\tilde{\boldsymbol{C}}}^{-1}\cdot\cdot\,\tilde{\sigma}=0$. Hence, $\tilde{\varepsilon}^*=\tilde{\varepsilon}(\boldsymbol{u})\in H_u$.

To the contrary, let $\tilde{\varepsilon}^*\in H_u$. In this case, we have $\tilde{\sigma}=\tilde{\tilde{\boldsymbol{C}}}\cdot\cdot\,(\nabla\boldsymbol{u}+\boldsymbol{u}\nabla)=\tilde{\tilde{\boldsymbol{C}}}\cdot\cdot\,\nabla\boldsymbol{z}$, where $\boldsymbol{u}$ is the real displacement, and displacement $\boldsymbol{v}$ corresponds to eigenstrain $\tilde{\varepsilon}^*$, $\boldsymbol{z}=\boldsymbol{u}-\boldsymbol{v}$.

From Eq. (3), it follows at $\boldsymbol{b}=0$ and $\boldsymbol{t}=0$ that

$$\int_\Omega\tilde{\tilde{\boldsymbol{C}}}\cdot\cdot\,\nabla\boldsymbol{z}\cdot\cdot\,\nabla\boldsymbol{w}\,\mathrm{d}V=0$$

$$\forall\,\boldsymbol{w}\in(W_2^1(\Omega))^3,\ \boldsymbol{w}=0,\ \boldsymbol{x}\in\Gamma_u$$

Let $\boldsymbol{w}=\boldsymbol{z}$, then we have

$$\int_\Omega\tilde{\tilde{\boldsymbol{C}}}\cdot\cdot\,\nabla\boldsymbol{z}\cdot\cdot\,\nabla\boldsymbol{z}\,\mathrm{d}V=0$$

Due to positive definiteness of matrix $\boldsymbol{C}_{ijkl}$, we can derive that $\boldsymbol{z}=0$ almost everywhere. As a result, $\tilde{\sigma}=0$.

Therefore, condition $\tilde{\varepsilon}^* \in H_u$ is necessary and sufficient one for the eigenstrain to be impotent.

It was shown[6] that a strain $\tilde{\varepsilon}$ is a compatible one ($\tilde{\varepsilon}^* \in H_u$), if and only if there exist such a (fictitious) body force $\boldsymbol{b} \in (L^2(\Omega))^3$, $\boldsymbol{x} \in \Omega$, and surface traction $\boldsymbol{t} \in (L^2(\Gamma_\sigma))^3$, $\boldsymbol{x} \in \Gamma_\sigma$, that produce in the elastic body a deformation with load strains equal to this eigenstrain without producing additional stress[4].

Another subspace H_σ is introduced under following condition: All elements of this subspace are nilpotent eigenstrains $\tilde{\varepsilon}_\sigma^*$.

For the generalized solution (Eqs. (2, 3)), we have for element $\tilde{\varepsilon}_\sigma^* \in H_\sigma$

$$\int_\Omega \tilde{\sigma} \cdot\cdot\, \tilde{\varepsilon}(\boldsymbol{w}) \mathrm{d}V = 0$$

$$\forall \boldsymbol{w} \in (W_2^1(\Omega))^3, \quad \boldsymbol{w}=0, \quad \boldsymbol{x} \in \Gamma_u \tag{14}$$

where $\tilde{\sigma} = -\tilde{\tilde{\boldsymbol{C}}} \cdot\cdot\, \tilde{\varepsilon}^*$.

5 Theorem on Decomposition of Eigenstrain

This theorem represents a general property of any eigenstrain tensor.

Theorem 1 Any tensor of eigenstrain $\tilde{\varepsilon}^* \in H$ can be decomposed uniquely into two orthogonal parts: Impotent and nilpotent constituents, namely

$$\tilde{\varepsilon} = \tilde{\varepsilon}_1^* + \tilde{\varepsilon}_2^* \tag{15}$$

where $\tilde{\varepsilon}_1^* \in H_u$ and $\tilde{\varepsilon}_2^* \in H_\sigma$.

Therefore, subspaces H_u and H_σ are mutually orthogonal (Fig. 1).

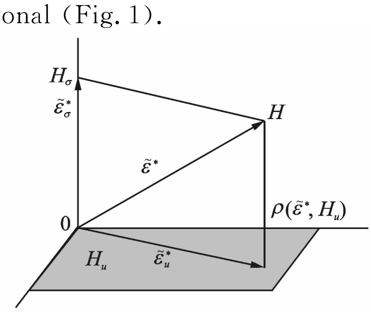

Fig. 1 Illustration to decomposition of eigenstrain in space H and orthogonality of subspaces H_u and H_σ

This theorem can be deduced from the theorem on orthogonal decomposition of Hilbert space H[5,7].

In this text, we prove that the decomposition is unique. To the contrary, we assume that decomposition Eq. (15) is not unique. Hence, we have another decomposition

$$\tilde{\varepsilon}^* = \tilde{v}_1^* + \tilde{v}_2^*$$

where $\tilde{v}_1^* \in H_1$, $\tilde{v}_2^* \in H_2$.

Then

$$\tilde{\varepsilon}_1^* + \tilde{\varepsilon}_2^* = \tilde{v}_1^* + \tilde{v}_2^*$$

$$\tilde{\varepsilon}_1^* - \tilde{v}_1^* = \tilde{v}_2^* - \tilde{\varepsilon}_2^*$$

Further, we introduce new designations

$$\tilde{\varepsilon}_1^* - \tilde{v}_1^* = \tilde{w}_1^* \in H_u, \ \tilde{v}_2^* - \tilde{\varepsilon}_2^* = \tilde{w}_2^* \in H_\sigma$$

As a result

$$\tilde{w}_1^* = \tilde{w}_2^*$$

The boundary value problems (Eqs. (9, 10)) indicate at once that the sole element which would belong to both subspaces H_u and H_σ must be the zero element.

Therefore

$$\tilde{w}_1^* = \tilde{w}_2^* = 0, \ \tilde{\varepsilon}_1^* = \tilde{v}_1^*, \ \tilde{\varepsilon}_2^* = \tilde{v}_2^*$$

and the decomposition in Eq. (15) turns to be unique.

As a consequence, the subspaces H_u and H_σ must be mutual orthogonal, i. e. any arbitrary tensor elements $\tilde{\varepsilon}_1^* \in H_u$ and $\tilde{\varepsilon}_2^* \in H_\sigma$ are orthogonal. Their scalar product is defined by

$$(\tilde{\varepsilon}_1^*, \tilde{\varepsilon}_2^*) = \int_\Omega \tilde{\varepsilon}_1^* \cdot\cdot\, \tilde{\tilde{\boldsymbol{C}}} \cdot\cdot\, \tilde{\varepsilon}_2^* \mathrm{d}V =$$

$$-\int_\Omega \tilde{\varepsilon}(\boldsymbol{u}) \cdot\cdot\, \tilde{\sigma} \mathrm{d}V = 0$$

It vanishes since the eigenstrains $\tilde{\varepsilon}_1^*$ and $\tilde{\varepsilon}_2^*$ satisfy Eqs. (8, 9) or Eq. (3) in the absence of external force ($\boldsymbol{b}=0$, $\boldsymbol{t}=0$). It is known that two subspaces H_u and H_σ in H are mutually orthogonal if any $\tilde{\varepsilon}_1^* \in H_u$ is orthogonal to any element of H_σ.

It can be concluded that there exists the orthogonal decomposition of the Hilbert space H in to subspaces H_u and H_σ ($H = H_u \oplus H_\sigma$). Consequently, the theorem on decomposition can be reformulated in another form.

Theorem 2 Subspaces H_u and H_σ are mutually orthogonal subspaces of Hilbert space H

($H=H_u \oplus H_\sigma$) and any element $\tilde{\varepsilon}^* \in H_u$ can be uniquely represented in the form of Eq. (15).

The orthogonality of the two subspaces allows us to prove the uniqueness of the decomposition in Eq. (15) by the other way. The scalar product $(\widetilde{w}_1^*, \widetilde{w}_2^*)$ vanishes.

Therefore, from $\widetilde{w}_1^* = \widetilde{w}_2^*$, we can obtain

$$(\widetilde{w}_1^*, \widetilde{w}_2^*) = (\widetilde{w}_1^*, \widetilde{w}_1^*) = 0 \Rightarrow$$
$$\widetilde{w}_1^{*\,2} = 0 \Rightarrow \widetilde{w}_1^* = \widetilde{w}_2^* = 0$$

and consequently the decomposition in Eq. (15) is unique.

Specifically, thermal strain can be decomposed in a unique way into impotent and nilpotent constituents. It is a novel result in linear thermoelasticity.

Decomposition of thermal strain opens the practically important opportunity to fully separate the control of strain and stress produced by force loading.

Two corollaries of practical importance are proven subsequently.

Corollary 1

Let $\tilde{\sigma}^o(\boldsymbol{x})$ be a statically admissible stress tensor. It satisfies Eq. (3) with $\boldsymbol{b}=0$ and $\boldsymbol{t}=0$. A tensor $\boldsymbol{f}$ is defined by

$$\boldsymbol{f} = \tilde{\varepsilon}^* + \tilde{\tilde{\boldsymbol{C}}}^{-1} \cdot\cdot\, \tilde{\sigma}^o$$

The condition $\boldsymbol{f} \in H_u$ is necessary and sufficient one in that stress $\tilde{\sigma}(\boldsymbol{x})$ is equal to its prescribed value $\tilde{\sigma}^o(\boldsymbol{x})$ in the region Ω.

Proof of necessity

Let $\tilde{\sigma}(\boldsymbol{x}) = \tilde{\sigma}^o(\boldsymbol{x})$. Take notice that $(-\tilde{\tilde{\boldsymbol{C}}}^{-1} \cdot\cdot\, \tilde{\sigma}) = \tilde{\varepsilon}_2^* \in H_\sigma$ in view of the fact that $\tilde{\sigma}^o$ is statically admissible and thus it equals the nilpotent part of $\tilde{\varepsilon}^*$, therefore $\boldsymbol{f} \in H_u$.

Proof of sufficiency

Let $\boldsymbol{f} \in H_u$ according to the decomposition theorem, and the tensor $(-\tilde{\tilde{\boldsymbol{C}}}^{-1} \cdot\cdot\, \tilde{\sigma}) \in H_\sigma$ is the nilpotent part of $\tilde{\varepsilon}^*$. Since decomposition is unique, we have

$$-\tilde{\tilde{\boldsymbol{C}}}^{-1} \cdot\cdot\, \tilde{\sigma} = -\tilde{\tilde{\boldsymbol{C}}}^{-1} \cdot\cdot\, \tilde{\sigma}^o$$
$$-\tilde{\tilde{\boldsymbol{C}}}^{-1} \cdot\cdot\, (\tilde{\sigma} - \tilde{\sigma}^o) = 0$$

From positive definiteness of the tensor $\tilde{\tilde{\boldsymbol{C}}}^{-1}$, it is concluded that $\tilde{\sigma} - \tilde{\sigma}^o = 0$, or $\tilde{\sigma} = \tilde{\sigma}^o$. Hence, the proof of Corollary 1 is completed. In particular, the condition $\tilde{\varepsilon}^* \in H_u$ is necessary and sufficient one for obtaining the stress-free state ($\tilde{\sigma}^o = 0$).

Corollary 2

This corollary is analogous to Corollary 1 except of non-vanishing external loads, $\boldsymbol{b} \neq 0$ and $\boldsymbol{t} \neq 0$. In this case, due to linearity of the generalized solution with respect to tensors $\tilde{\sigma}$ and $\bar{\varepsilon}$, we can represent the solution in the form

$$\tilde{\sigma} = \tilde{\sigma}^{\mathrm{F}} + \tilde{\sigma}^{\mathrm{e}}, \qquad \tilde{\varepsilon} = \tilde{\varepsilon}^{\mathrm{F}} + \tilde{\varepsilon}^{\mathrm{e}} \tag{16}$$

where $\tilde{\sigma}^{\mathrm{F}}$ and $\bar{\varepsilon}^{\mathrm{F}}$ are tensors of stress and strain due to imposed volume forces $\boldsymbol{b}$ and surface traction $\boldsymbol{t}$, $\tilde{\sigma}^{\mathrm{e}}$ and $\tilde{\varepsilon}^{\mathrm{e}}$ are tensors of stress and strain due to eigenstrain. Therefore, Corollary 1 is valid for this case when $(\tilde{\sigma}^u - \tilde{\sigma}^{\mathrm{F}})$ and $(\tilde{\varepsilon}^u - \tilde{\varepsilon}^{\mathrm{F}})$ are substituted for $\tilde{\sigma}^o$ and $\tilde{\varepsilon}^o$.

Thus, Corollaries 1 and 2 provide the necessary and sufficient conditions for obtaining a prescribed stress state by means of imposed eigenstrain (in particular, thermal strain).

The introduction of subspace H_u transfers the classical conditions of strain compatibility to the membership of an element of space H in the subspace H_u. It is very important that measure of strain incompatibility can be defined (note that the total strain tensor is always compatible whereas its constituents can be incompatible). Furthermore, it is significant that the classical equations of compatibility demand the existence of the second spatial derivatives of components of the strain tensor. In the above definition, this condition is not imposed. The measure of tensor incompatibility can be introduced as a distance $\rho(\tilde{\varepsilon}^*, H_u)$ of given tensor to subspace H_u. This measure can be found through the basis in subspace H_u (or subspace H_σ).

It is possible to prove that distance $\rho(\tilde{\varepsilon}^*, H_u)$ is related to level of thermal stresses in a body[7]. For this aim, it is useful to introduce the objective function of the problem obtaining given eigenstress (in particular, thermal stress) $\tilde{\rho}^o$.

Namely

$$\psi=\tilde{\rho}^{o}-\tilde{\rho}_{H}^{2}=\int_{\Omega}(\tilde{\rho}^{o}-\tilde{\rho})\cdot\cdot\,\tilde{C}^{-1}\cdot\cdot\,(\tilde{\rho}^{o}-\tilde{\rho})\,\mathrm{d}V\rightarrow$$
$$\rightarrow\inf(\tilde{\varepsilon}_{\sigma}^{*})\qquad(17)$$

where $\tilde{\rho}=-\tilde{C}^{-1}\cdot\cdot\,\tilde{\varepsilon}_{\sigma}^{*}$, $\tilde{\rho}_{o}=-\tilde{C}^{-1}\cdot\cdot\,\tilde{\varepsilon}_{\sigma_{o}}^{*}$.

Eq. (17) can be written in another form

$$\psi=\int_{\Omega}(\tilde{\varepsilon}_{\sigma}^{*}-\tilde{\varepsilon}_{\sigma_{o}}^{*})\cdot\cdot\,\tilde{C}\cdot\cdot\,(\tilde{\varepsilon}_{\sigma}^{*}-\tilde{\varepsilon}_{\sigma_{o}}^{*})\,\mathrm{d}V\qquad(18)$$

The examples of application of proposed approach to control of eigenstress and eigenstrain are discussed in another presentation.

References:

[1] Reissner H. Eigenspannungen und eigenspannungsquellen[J]. Z Angew Math Mech, 1931, 11(1): 1-8.

[2] Duvant J, Lions T. Les inequations en mechanique et en physique[M]. Paris: Dunod, 1972.

[3] Mura T. Micromechanics of defects in solids[M]. 2nd edition. Kluwer, Dordrecht: Springer, 1991.

[4] Irschik H, Ziegler F. Eigenstrain without stress and static shape control of structures[J]. AIAA Journal, 2001, 39(10): 1985.

[5] Lebedev L P, Vorovich I I. Functional analysis in mechanics[M]. [S. l.]: Springer-Verlag, 2000.

[6] Nyashin Y, Kirykhin V. Biological stresses in living tissues: The modeling and control problem[J]. Russian Journal of Biomechanics, 2002, 6(3): 13.

[7] Nyashin Y, Lokhov V, Ziegler F. Decomposition method in linear elastic problems with eigenstrain [J]. Z Angew Math Mech, 2005, 65: 557.

(Executive editor: Zhang Huangqun)