Coupled FEM Analyses on Electro-mechanical Problem of Electrostrictive Plate with Elliptical Hole

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Abstract: A detailed theoretical construction of general coupled 3D FEM analyses of anisotropic dielectrics is first presented by considering the electric body force and body couple moment. A 3D electrostrictive element is subsequently defined in ABAQUS user subroutine UEL and the post-processing of finite element method (FEM) results is realized by UVARM and dummy element method. Then the developed technique is used to solve the electro-elastic field of an isotropic electrostrictive plate with an elliptical hole subjected to electrical load. By comparing the coupled and uncoupled numerical results, the traditional uncoupled analytical method can cause a large error when the applied electric field or the electrostrictive performance of the dielectric is high, and thus the present coupled analysis is especially necessary.

Key words: electrostrictive dielectric; anisotropy; FEM; UEL; elliptical hole

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1 Introduction

Due to smaller hysteresis and lower aging effect than piezoelectric materials, electrostrictive dielectrics have attracted much research interest[1-5]. Currently, most of the work studying the electro-mechanical behavior of an electrostrictive plate with an elliptical hole, such as Knops[6], Smith and Warren[7], Jiang and Kuang[8], and Gao, et al.[9], focuses on the two-dimensional infinite isotropic problem without considering the electric body couple moment, which is called as the uncoupled analytical method, i. e., neglecting the effect of strain-stress field on the electric filed.

Although much of the FEM work has been done on smart materials[10-12], such as piezoelectric and ferroelectric materials, little work has been done on the case of finite electrostrictive plates. Recently, Gil and Ledger[13] provided a coupled hp-finite element scheme for the solution of 2D electrostrictive materials, and then Jin, et al.[14] extended this method to solve the 2D problem of electrostrictive and magnetostrictive materials. In the traditional purely elastic 2D generalized plane stress problem, it is assumed that the average value of stress σ_{33} along the thickness x_3 is zero (Lekhnitskii[15]). However, when the electric body fore $\boldsymbol{f}_M$, electric body couple moment $\boldsymbol{m}_M$, and Maxwell stress $\boldsymbol{\sigma}_M$ are taken into consideration, whether can we still take this assumption or suppose the average value of total stress σ_{eq33} $(=\sigma_{33}+\sigma_{\mathrm{M33}})$ to be zero?

This paper mainly contributes three aspects. First, a quite different 3D finite element method (FEM) analysis based on ABAQUS user subroutines UEL, UVARM and dummy element method is developed. Second, the problem is extended to a 3D anisotropic case, and the electric body

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force, electric body couple moment, the asymmetry of Cauchy stress tensor and Maxwell stress tensor are also considered, providing a more general coupled and uncoupled FEM construction. Third, the uncoupled and coupled FEM results are compared, verifying the traditional uncoupled method.

2 Basic Equations

As shown in Fig. 1, a finite anisotropic electrostrictive plate with an elliptical hole is subjected to electrical potential ϕ_{up}, ϕ_{down} and mechanical load $\boldsymbol{p}_b$ at its boundaries.

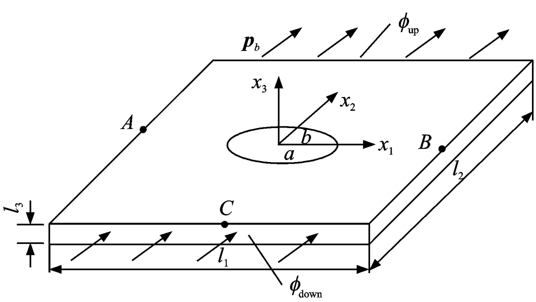


Fig. 1 Finite anisotropic electrostrictive plate with an elliptical hole subjected to electrical and mechanical loads

The basic differential equations describing the anisotropic electrostrictive dielectric, where the body force and body couple moment are both taken into consideration, are given by

(1) Maxwell Equations[16]

$$\nabla \cdot \boldsymbol{D}=0,\ \nabla \times \boldsymbol{E}=\boldsymbol{0} \Rightarrow \boldsymbol{E}=-\nabla \otimes \phi \quad (1)$$

where $\boldsymbol{E}$, $\boldsymbol{D}$, ϕ are the electric field, electric displacement and electric potential, respectively.

(2) Equilibrium equations

$$\nabla \cdot \boldsymbol{\sigma}^{\mathrm{T}}+\boldsymbol{f}_M=\nabla \cdot \boldsymbol{\sigma}^{\mathrm{T}}+\nabla \cdot \boldsymbol{\sigma}_M=\boldsymbol{0} \quad (2)$$

$$\in : \boldsymbol{\sigma}^{\mathrm{T}}+\boldsymbol{m}_M=\boldsymbol{0} \Rightarrow \frac{\boldsymbol{\sigma}^{\mathrm{T}}-\boldsymbol{\sigma}}{2}=\frac{1}{2}\boldsymbol{h}_M \quad (3)$$

where $\boldsymbol{\sigma}$, $\boldsymbol{f}_M$, $\boldsymbol{\sigma}_M$ and $\boldsymbol{m}_M$ are the Cauchy stress tensor, electric body force, Maxwell stress tensor and electric body couple moment.

$$\boldsymbol{\sigma}_M=\boldsymbol{E} \otimes \boldsymbol{D}-\frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D})\boldsymbol{I} \quad (4)$$

$$\boldsymbol{m}_M=\boldsymbol{P} \times \boldsymbol{E},\ \boldsymbol{h}_M=\boldsymbol{E} \otimes \boldsymbol{P}-\boldsymbol{P} \otimes \boldsymbol{E} \quad (5)$$

It is obvious from Eqs. (3,4) that the Cauchy stress tensor $\boldsymbol{\sigma}$ and Maxwell stress tensor $\boldsymbol{\sigma}_M$ are not symmetric in general.

(3) Deformation equation

$$\boldsymbol{\varepsilon}=\frac{1}{2}(\boldsymbol{u} \otimes \nabla+\nabla \otimes \boldsymbol{u}) \quad (6)$$

where $\boldsymbol{\varepsilon}$ is the strain tensor and $\boldsymbol{u}$ the displacement vector.

(4) Constitutive equations

$$\frac{1}{2}(\sigma_{ij}+\sigma_{ji})=\rho_{\mathrm{mass}}\left(C_{ijmn}\varepsilon_{mn}+\frac{1}{2}\eta_{ijmn}E_m E_n\right) \quad (7)$$

$$P_m=-\rho_{\mathrm{mass}}\zeta_{mk}E_k-\rho_{\mathrm{mass}}\eta_{ijmn}\varepsilon_{ij}E_n \quad (8)$$

where ρ_{mass}, ζ_{ij}, C_{ijmn}, and η_{ijmn} are the mass density, polarization coefficients, linear elastic coefficients, and electrostrictive coefficients, respectively, which satisfy

$$\zeta_{ij}=\zeta_{ji},\ C_{ijmn}=C_{ijnm}=C_{jimn}=C_{mnij}$$
$$\eta_{ijmn}=\eta_{ijnm}=\eta_{jimn} \quad (9)$$

The constitutive equations are obtained by extending the work of Kuang[17-19] where we consider the electric body couple moment and asymmetry of Cauchy stress tensor; and relation is derived by taking the method of Ting[20] when he deals with purely anisotropic elasticity.

(5) FEM conditions

$$\boldsymbol{\sigma} \cdot \boldsymbol{n}=\boldsymbol{p}_b-(\boldsymbol{\sigma}_M)^{\mathrm{T}} \cdot \boldsymbol{n},\ \boldsymbol{D} \cdot \boldsymbol{n}=\sigma_b,\ \boldsymbol{u}=\boldsymbol{u}_b,\ \phi=\phi_b \quad (10)$$

where $\boldsymbol{p}_b$ is the mechanical force, and σ_b the net free surface charge density. Eq. (10) means that the electric field is applied on the electrostrictive dielectric directly, rather than on the environment (vacuum or air), therefore the Maxwell stress contributed by the environment is then equal to zero.

3 Virtual Displacement Principle

Under small deformation assumption, according to the principle of virtual work, one has the following equation

$$\iiint_{\Omega}[(\nabla \cdot \boldsymbol{\sigma}^{\mathrm{T}}+\boldsymbol{f}_M) \cdot \delta\boldsymbol{u}+(\in : \boldsymbol{\sigma}^{\mathrm{T}}+\boldsymbol{m}_M) \cdot \delta\boldsymbol{\omega}+(\nabla \cdot \boldsymbol{D})\delta\phi]\,\mathrm{d}\Omega=\iint_{S_b}(\boldsymbol{\sigma} \cdot \boldsymbol{n}-\boldsymbol{p}_b+\boldsymbol{\sigma}_M \cdot \boldsymbol{n}) \cdot \delta\boldsymbol{u}\,\mathrm{d}S+\iint_{S_b}(\boldsymbol{D} \cdot \boldsymbol{n}-\sigma_b)\delta\phi\,\mathrm{d}S \quad (11)$$

where Ω is a three dimensional volume bounded by surface S_b; $\mathrm{d}\boldsymbol{S}=\boldsymbol{n} \cdot \mathrm{d}S$, and $\boldsymbol{n}$ the outward normal vector at area element dS. And

$$\boldsymbol{\omega} = -\frac{1}{2} \in : \boldsymbol{w},\ \boldsymbol{w} = \frac{1}{2}(\boldsymbol{u} \otimes \nabla - \nabla \otimes \boldsymbol{u}) \quad (12)$$

Using the following equations

$$\delta \boldsymbol{w} : \frac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{T}}) = (\in : \boldsymbol{\sigma}^{\mathrm{T}}) \cdot \delta \boldsymbol{\omega} \quad (13)$$

$$\boldsymbol{m}_M \cdot \delta \boldsymbol{\omega} = \left(-\frac{1}{2} \in : \boldsymbol{h}_M\right) \cdot \delta \boldsymbol{\omega} = \frac{1}{2} \delta \boldsymbol{w} : \boldsymbol{h}_M \quad (14)$$

Eq. (11) can be simplified as

$$\iiint_\Omega \boldsymbol{\sigma}_{\mathrm{eqsy}} : \delta \varepsilon \, \mathrm{d}\Omega - \iiint_\Omega \boldsymbol{\sigma}_{\mathrm{eqan}} : \delta \boldsymbol{w} \, \mathrm{d}\Omega - \iiint_\Omega \boldsymbol{D} \cdot (\delta \boldsymbol{E}) \, \mathrm{d}\Omega = \iint_{S_b} (\boldsymbol{p}_b - (\boldsymbol{\sigma}_M)^{\mathrm{T}} \cdot \boldsymbol{n}) \cdot \delta \boldsymbol{u} \, \mathrm{d}S + \oiint_S \delta \boldsymbol{u} \cdot (\boldsymbol{\sigma}_M)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{S} + \iint_{S_b} \sigma_b \delta \phi \, \mathrm{d}S = \iint_{S_b} \boldsymbol{p}_b \cdot \delta \boldsymbol{u} \, \mathrm{d}S + \iint_{S_b} \sigma_b \delta \phi \, \mathrm{d}S \quad (15)$$

where

$$\boldsymbol{\sigma}_{\mathrm{eqsy}} = \frac{1}{2}[(\boldsymbol{\sigma}^{\mathrm{T}} + \boldsymbol{\sigma}_M) + (\boldsymbol{\sigma}^{\mathrm{T}} + \boldsymbol{\sigma}_M)^{\mathrm{T}}] = \frac{\boldsymbol{\sigma}^{\mathrm{T}} + \boldsymbol{\sigma}}{2} + \frac{1}{2}[\boldsymbol{E} \otimes \boldsymbol{D} + (\boldsymbol{E} \otimes \boldsymbol{D})^{\mathrm{T}}] - \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D}) \boldsymbol{I} \quad (16)$$

$$\boldsymbol{\sigma}_{\mathrm{eqan}} = \frac{1}{2}[(\boldsymbol{\sigma}^{\mathrm{T}} + \boldsymbol{\sigma}_M) - (\boldsymbol{\sigma}^{\mathrm{T}} + \boldsymbol{\sigma}_M)^{\mathrm{T}}] = \boldsymbol{E} \otimes \boldsymbol{P} - \boldsymbol{P} \otimes \boldsymbol{E} = \boldsymbol{h}_M \quad (17)$$

In fact there is another way to obtain Eq. (15). By introducing Eqs. (16, 17), one may obtain the equivalent expressions of Eqs. (2, 3, 7, 8, 10), that is,

$$\nabla \cdot \boldsymbol{\sigma}_{\mathrm{eqsy}} + \nabla \cdot \boldsymbol{\sigma}_{\mathrm{eqan}} = \boldsymbol{0} \quad (18)$$

$$D_m = (\varepsilon_0 \delta_{mk} - \rho_{\mathrm{mass}} \zeta_{mk}) E_k - \rho_{\mathrm{mass}} \eta_{ijmn} \varepsilon_{ij} E_n \quad (19)$$

$$\sigma_{\mathrm{eqsy}\,ij} = \rho_{\mathrm{mass}} C_{ijmn} \varepsilon_{mn} + \frac{1}{2} \rho_{\mathrm{mass}} \eta_{ijmn} E_m E_n + \frac{1}{2}(E_i D_j + D_i E_j) - \frac{1}{2} E_k D_k \delta_{ij} \quad (20)$$

$$\boldsymbol{\sigma}_{\mathrm{eqsy}} \cdot \boldsymbol{n} = \boldsymbol{p}_b + \boldsymbol{\sigma}_{\mathrm{eqan}} \cdot \boldsymbol{n},\ \boldsymbol{D} \cdot \boldsymbol{n} = \sigma_b,\ \boldsymbol{u} = \boldsymbol{u}_b,\ \phi = \phi_b \quad (21)$$

Then the principle of virtual displacement gives

$$\iiint_\Omega [(\nabla \cdot \boldsymbol{\sigma}_{\mathrm{eqsy}} + \nabla \cdot \boldsymbol{\sigma}_{\mathrm{eqan}}) \cdot \delta \boldsymbol{u} + (\nabla \cdot \boldsymbol{D}) \delta \phi] \, \mathrm{d}\Omega = \iint_{S_b} (\boldsymbol{\sigma}_{\mathrm{eqsy}} \cdot \boldsymbol{n} - \boldsymbol{p}_b - \boldsymbol{\sigma}_{\mathrm{eqan}} \cdot \boldsymbol{n}) \cdot \delta \boldsymbol{u} \, \mathrm{d}S + \iint_{S_b} (\boldsymbol{D} \cdot \boldsymbol{n} - \sigma_b) \delta \phi \, \mathrm{d}S \quad (22)$$

Since

$$\iiint_\Omega [(\nabla \cdot \boldsymbol{\sigma}_{\mathrm{eqsy}} + \nabla \cdot \boldsymbol{\sigma}_{\mathrm{eqan}}) \cdot \delta \boldsymbol{u} + (\nabla \cdot \boldsymbol{D}) \delta \phi] \mathrm{d}\Omega = \iiint_\Omega [\nabla \cdot (\boldsymbol{\sigma}_{\mathrm{eqsy}} \cdot \delta \boldsymbol{u}) - \boldsymbol{\sigma}_{\mathrm{eqsy}} : (\nabla \otimes \delta \boldsymbol{u})] \mathrm{d}\Omega + \iiint_\Omega [\nabla \cdot (\boldsymbol{D} \delta \phi) - \boldsymbol{D} \cdot (\nabla \otimes \delta \phi)] \mathrm{d}\Omega + \iiint_\Omega [\nabla \cdot (\boldsymbol{\sigma}_{\mathrm{eqan}} \cdot \delta \boldsymbol{u}) - \boldsymbol{\sigma}_{\mathrm{eqan}} : (\nabla \otimes \delta \boldsymbol{u})] \mathrm{d}\Omega = \iint_{S_b} \mathrm{d}\boldsymbol{S} \cdot (\boldsymbol{\sigma}_{\mathrm{eqsy}} \cdot \delta \boldsymbol{u}) - \iiint_\Omega \boldsymbol{\sigma}_{\mathrm{eqsy}} : \delta \varepsilon \, \mathrm{d}\Omega + \iint_{S_b} \mathrm{d}\boldsymbol{S} \cdot (\boldsymbol{D} \delta \phi) + \iiint_\Omega \boldsymbol{D} \cdot (\delta \boldsymbol{E}) \, \mathrm{d}\Omega + \iint_{S_b} \mathrm{d}\boldsymbol{S} \cdot (\boldsymbol{\sigma}_{\mathrm{eqan}} \cdot \delta \boldsymbol{u}) + \iiint_\Omega \boldsymbol{\sigma}_{\mathrm{eqan}} : \delta \boldsymbol{w} \, \mathrm{d}\Omega = \iint_{S_b} \delta \boldsymbol{u} \cdot (\boldsymbol{\sigma}_{\mathrm{eqsy}})^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{S} + \iint_{S_b} \delta \phi \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} + \iint_{S_{b\sigma}} \delta \boldsymbol{u} \cdot (\boldsymbol{\sigma}_{\mathrm{eqan}})^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{S} - \iiint_\Omega \boldsymbol{\sigma}_{\mathrm{eqsy}} : \delta \boldsymbol{\varepsilon} \, \mathrm{d}\Omega + \iiint_\Omega \boldsymbol{D} \cdot (\delta \boldsymbol{E}) \, \mathrm{d}\Omega + \iiint_\Omega \boldsymbol{\sigma}_{\mathrm{eqan}} : \delta \boldsymbol{w} \, \mathrm{d}\Omega \quad (23)$$

Inserting Eq. (23) into Eq. (22), one may obtain the same equation as Eq. (15), where

$$\iiint_\Omega \boldsymbol{\sigma}_{\mathrm{eqsy}} : \delta \boldsymbol{\varepsilon} \, \mathrm{d}\Omega - \iiint_\Omega \boldsymbol{\sigma}_{\mathrm{eqan}} : \delta \boldsymbol{w} \, \mathrm{d}\Omega - \iiint_\Omega \boldsymbol{D} \cdot \delta \boldsymbol{E} \, \mathrm{d}\Omega = \iint_{S_b} \boldsymbol{p}_b \cdot \delta \boldsymbol{u} \, \mathrm{d}S + \iint_{S_b} \sigma_b \delta \phi \, \mathrm{d}S \quad (24)$$

4 Matrix Representation

In Sections 2−3, we use bold symbols for vectors (first-order tensors) and second-order tensors. This convention holds for the whole paper. As to matrix, we add a "=" to distinguish them from tensors.

Define

$$\underline{\underline{u}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},\ \underline{\underline{E}} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix},\ \underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}$$

$$\underline{\underline{w}} = \begin{bmatrix} 2w_{23} \\ 2w_{13} \\ 2w_{12} \end{bmatrix},\ \underline{\underline{\gamma}} = \begin{bmatrix} E_1 E_1 \\ E_2 E_2 \\ E_3 E_3 \\ 2E_2 E_3 \\ 2E_1 E_3 \\ 2E_1 E_2 \end{bmatrix} \quad (25)$$

and

$$\underline{\boldsymbol{\sigma}}_{\mathrm{eqsy}}=\begin{bmatrix}\sigma_{sy11}\\ \sigma_{sy22}\\ \sigma_{sy33}\\ \sigma_{sy23}\\ \sigma_{sy13}\\ \sigma_{sy12}\end{bmatrix},\ \underline{\boldsymbol{\sigma}}_{\mathrm{eqan}}=\begin{bmatrix}\sigma_{an23}\\ \sigma_{an13}\\ \sigma_{an12}\end{bmatrix}$$

$$\underline{\boldsymbol{D}}=\begin{bmatrix}D_1\\ D_2\\ D_3\end{bmatrix},\ \underline{\boldsymbol{p}}_b=\begin{bmatrix}p_{b1}\\ p_{b2}\\ p_{b3}\end{bmatrix} \tag{26}$$

then one has from Eqs. (6,19,20) that

$$\underline{\boldsymbol{E}}=\underline{\underline{\boldsymbol{L}}}_{\phi}\phi,\ \underline{\boldsymbol{\gamma}}=\underline{\underline{\boldsymbol{B}}}_E\underline{\boldsymbol{E}},\ \underline{\boldsymbol{\varepsilon}}=\underline{\underline{\boldsymbol{L}}}_{syu}\underline{\boldsymbol{u}},\ \underline{\boldsymbol{w}}=\underline{\underline{\boldsymbol{L}}}_{anu}\underline{\boldsymbol{u}} \tag{27}$$

and

$$\underline{\boldsymbol{D}}=\underline{\underline{\boldsymbol{\alpha}}}_{eq}\underline{\boldsymbol{E}} \tag{28}$$

$$\underline{\boldsymbol{\sigma}}_{\mathrm{eqan}}=\underline{\underline{\boldsymbol{\lambda}}}_{eq}\underline{\boldsymbol{\gamma}}=\underline{\underline{\boldsymbol{\lambda}}}_{eq}\underline{\underline{\boldsymbol{B}}}_E\underline{\boldsymbol{E}} \tag{29}$$

$$\underline{\boldsymbol{\sigma}}_{\mathrm{eqsy}}=\underline{\underline{\boldsymbol{C}}}_{eq}\underline{\boldsymbol{\varepsilon}}+\underline{\underline{\boldsymbol{\beta}}}_{eq}\underline{\boldsymbol{\gamma}}=\underline{\underline{\boldsymbol{C}}}_{eq}\underline{\underline{\boldsymbol{L}}}_{syu}\underline{\boldsymbol{u}}+\underline{\underline{\boldsymbol{\beta}}}_{eq}\underline{\underline{\boldsymbol{B}}}_E\underline{\boldsymbol{E}} \tag{30}$$

where

$$\underline{\underline{\boldsymbol{L}}}_{\phi}=\begin{bmatrix}-\dfrac{\partial}{\partial x_1}\\ -\dfrac{\partial}{\partial x_2}\\ -\dfrac{\partial}{\partial x_3}\end{bmatrix},\ \underline{\underline{\boldsymbol{B}}}_E=\begin{bmatrix}E_1 & 0 & 0\\ 0 & E_2 & 0\\ 0 & 0 & E_3\\ 0 & E_3 & E_2\\ E_3 & 0 & E_1\\ E_2 & E_1 & 0\end{bmatrix} \tag{31}$$

$$\underline{\underline{\boldsymbol{L}}}_{syu}=\begin{bmatrix}\dfrac{\partial}{\partial x_1} & 0 & 0\\ 0 & \dfrac{\partial}{\partial x_2} & 0\\ 0 & 0 & \dfrac{\partial}{\partial x_3}\\ 0 & \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_2}\\ \dfrac{\partial}{\partial x_3} & 0 & \dfrac{\partial}{\partial x_1}\\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} & 0\end{bmatrix}$$

$$\underline{\underline{\boldsymbol{L}}}_{anu}=\begin{bmatrix}0 & \dfrac{\partial}{\partial x_3} & -\dfrac{\partial}{\partial x_2}\\ \dfrac{\partial}{\partial x_3} & 0 & -\dfrac{\partial}{\partial x_1}\\ \dfrac{\partial}{\partial x_2} & -\dfrac{\partial}{\partial x_1} & 0\end{bmatrix} \tag{32}$$

$$\underline{\underline{\boldsymbol{C}}}_{eq}=\begin{bmatrix}C_{eq11} & C_{eq12} & C_{eq13} & C_{eq14} & C_{eq15} & C_{eq16}\\ C_{eq12} & C_{eq22} & C_{eq23} & C_{eq24} & C_{eq25} & C_{eq26}\\ C_{eq13} & C_{eq23} & C_{eq33} & C_{eq34} & C_{eq35} & C_{eq36}\\ C_{eq14} & C_{eq24} & C_{eq34} & C_{eq44} & C_{eq45} & C_{eq46}\\ C_{eq15} & C_{eq25} & C_{eq35} & C_{eq45} & C_{eq55} & C_{eq56}\\ C_{eq16} & C_{eq26} & C_{eq36} & C_{eq46} & C_{eq56} & C_{eq66}\end{bmatrix} \tag{33}$$

$$\underline{\underline{\boldsymbol{\eta}}}_{eq}=\begin{bmatrix}\eta_{eq11} & \eta_{eq12} & \eta_{eq13} & \eta_{eq14} & \eta_{eq15} & \eta_{eq16}\\ \eta_{eq21} & \eta_{eq22} & \eta_{eq23} & \eta_{eq24} & \eta_{eq25} & \eta_{eq26}\\ \eta_{eq31} & \eta_{eq32} & \eta_{eq33} & \eta_{eq34} & \eta_{eq35} & \eta_{eq36}\\ \eta_{eq41} & \eta_{eq42} & \eta_{eq43} & \eta_{eq44} & \eta_{eq45} & \eta_{eq46}\\ \eta_{eq51} & \eta_{eq52} & \eta_{eq53} & \eta_{eq54} & \eta_{eq55} & \eta_{eq56}\\ \eta_{eq61} & \eta_{eq62} & \eta_{eq63} & \eta_{eq64} & \eta_{eq65} & \eta_{eq66}\end{bmatrix} \tag{34}$$

where the following index transformation in Table 1 is used for Eqs. (33, 34).

Table 1 Index transformation for fourth-order tensors

ij(or mn)	11	22	33	23 or 32	31 or 13	12 or 21
α(or β)	1	2	3	4	5	6

e. g. , $C_{eq12}=\rho_{\mathrm{mass}}C_{1122}$, $\eta_{eq54}=\rho_{\mathrm{mass}}\ \eta_{3123}=\rho_{\mathrm{mass}}\ \eta_{3132}=\rho_{\mathrm{mass}}\ \eta_{1223}=\rho_{\mathrm{mass}}\ \eta_{1232}$

The other quantities in Eqs. (28 — 30) are given by

$$\underline{\underline{\boldsymbol{\alpha}}}_{eq}=\underline{\underline{\boldsymbol{\alpha}}}_{\mathrm{one}}+\underline{\underline{\boldsymbol{\alpha}}}_{\mathrm{two}},\ \underline{\underline{\boldsymbol{\beta}}}_{eq}=\frac{1}{2}\underline{\underline{\boldsymbol{\eta}}}_{eq}+\underline{\underline{\boldsymbol{\beta}}}_{\mathrm{one}}+\underline{\underline{\boldsymbol{\beta}}}_{\mathrm{two}}$$

$$\underline{\underline{\boldsymbol{\lambda}}}_{eq}=\underline{\underline{\boldsymbol{\lambda}}}_{\mathrm{one}}+\underline{\underline{\boldsymbol{\alpha}}}_{\mathrm{two}} \tag{35}$$

where

$$\underline{\underline{\boldsymbol{\alpha}}}_{\mathrm{one}}=\begin{bmatrix}\varepsilon_0 & 0 & 0\\ 0 & \varepsilon_0 & 0\\ 0 & 0 & \varepsilon_0\end{bmatrix}+\underline{\underline{\boldsymbol{\zeta}}}_{eq}$$

$$\underline{\underline{\boldsymbol{\zeta}}}_{eq}=\begin{bmatrix}\zeta_{eq11} & \zeta_{eq12} & \zeta_{eq13}\\ \zeta_{eq12} & \zeta_{eq22} & \zeta_{eq23}\\ \zeta_{eq13} & \zeta_{eq23} & \zeta_{eq33}\end{bmatrix}$$

$$\zeta_{eqij}=-\rho_{\mathrm{mass}}\zeta_{ij} \tag{36}$$

$$\begin{bmatrix}a_{11}\\ a_{22}\\ a_{33}\\ a_{23}\\ a_{13}\\ a_{12}\end{bmatrix}=-\underline{\underline{\boldsymbol{\eta}}}_{eq}^{\mathrm{T}}\underline{\boldsymbol{\varepsilon}}=-\begin{bmatrix}\rho_{\mathrm{mass}}\eta_{ij11}\varepsilon_{ij}\\ \rho_{\mathrm{mass}}\eta_{ij22}\varepsilon_{ij}\\ \rho_{\mathrm{mass}}\eta_{ij33}\varepsilon_{ij}\\ \rho_{\mathrm{mass}}\eta_{ij23}\varepsilon_{ij}\\ \rho_{\mathrm{mass}}\eta_{ij13}\varepsilon_{ij}\\ \rho_{\mathrm{mass}}\eta_{ij12}\varepsilon_{ij}\end{bmatrix}$$

$$\underline{\underline{\boldsymbol{\alpha}}}_{\mathrm{two}}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{12} & a_{22} & a_{23}\\ a_{13} & a_{23} & a_{33}\end{bmatrix},\ a_{ij}=a_{ji} \tag{37}$$

$$\underline{\underline{\boldsymbol{\lambda}_{\text{one}}}}=\begin{bmatrix} 0 & \zeta_{eq23} & -\zeta_{eq23} & \frac{1}{2}(\zeta_{eq33}-\zeta_{eq22}) & -\frac{1}{2}\zeta_{eq12} & \frac{1}{2}\zeta_{eq13} \\ \zeta_{eq13} & 0 & -\zeta_{eq13} & -\frac{1}{2}\zeta_{eq12} & \frac{1}{2}(\zeta_{eq33}-\zeta_{eq11}) & \frac{1}{2}\zeta_{eq23} \\ \zeta_{eq12} & -\zeta_{eq12} & 0 & -\frac{1}{2}\zeta_{eq13} & \frac{1}{2}\zeta_{eq23} & \frac{1}{2}(\zeta_{eq22}-\zeta_{eq11}) \end{bmatrix} \tag{38}$$

$$\underline{\underline{\boldsymbol{\lambda}_{\text{two}}}}=\begin{bmatrix} 0 & a_{23} & -a_{23} & \frac{1}{2}(a_{33}-a_{22}) & -\frac{1}{2}a_{12} & \frac{1}{2}a_{13} \\ a_{13} & 0 & -a_{13} & -\frac{1}{2}a_{12} & \frac{1}{2}(a_{33}-a_{11}) & \frac{1}{2}a_{23} \\ a_{12} & -a_{12} & 0 & -\frac{1}{2}a_{13} & \frac{1}{2}a_{23} & \frac{1}{2}(a_{22}-a_{11}) \end{bmatrix} \tag{39}$$

$$\underline{\underline{\boldsymbol{\beta}_{\text{one}}}}=\begin{bmatrix} \frac{1}{2}\varepsilon_0+\frac{1}{2}\zeta_{eq11} & -\frac{1}{2}\varepsilon_0-\frac{1}{2}\zeta_{eq22} & -\frac{1}{2}\varepsilon_0-\frac{1}{2}\zeta_{eq33} & -\frac{1}{2}\zeta_{eq23} & 0 & 0 \\ -\frac{1}{2}\varepsilon_0-\frac{1}{2}\zeta_{eq11} & \frac{1}{2}\varepsilon_0+\frac{1}{2}\zeta_{eq22} & -\frac{1}{2}\varepsilon_0-\frac{1}{2}\zeta_{eq33} & 0 & -\frac{1}{2}\zeta_{eq13} & 0 \\ -\frac{1}{2}\varepsilon_0-\frac{1}{2}\zeta_{eq11} & -\frac{1}{2}\varepsilon_0-\frac{1}{2}\zeta_{eq22} & \frac{1}{2}\varepsilon_0+\frac{1}{2}\zeta_{eq33} & 0 & 0 & -\frac{1}{2}\zeta_{eq12} \\ 0 & \frac{1}{2}\zeta_{eq23} & \frac{1}{2}\zeta_{eq23} & \begin{pmatrix}\frac{1}{2}\varepsilon_0+\frac{1}{4}\zeta_{eq33} \\ +\frac{1}{4}\zeta_{eq22}\end{pmatrix} & \frac{1}{4}\zeta_{eq12} & \frac{1}{4}\zeta_{eq13} \\ \frac{1}{2}\zeta_{eq13} & 0 & \frac{1}{2}\zeta_{eq13} & \frac{1}{4}\zeta_{eq12} & \begin{pmatrix}\frac{1}{2}\varepsilon_0+\frac{1}{4}\zeta_{eq11} \\ +\frac{1}{4}\zeta_{eq33}\end{pmatrix} & \frac{1}{4}\zeta_{eq23} \\ \frac{1}{2}\zeta_{eq12} & \frac{1}{2}\zeta_{eq12} & 0 & \frac{1}{4}\zeta_{eq13} & \frac{1}{4}\zeta_{eq23} & \begin{pmatrix}\frac{1}{2}\varepsilon_0+\frac{1}{4}\zeta_{eq11} \\ +\frac{1}{4}\zeta_{eq22}\end{pmatrix} \end{bmatrix} \tag{40}$$

$$\underline{\underline{\boldsymbol{\beta}_{\text{two}}}}=\begin{bmatrix} \frac{1}{2}a_{11} & -\frac{1}{2}a_{22} & -\frac{1}{2}a_{33} & -\frac{1}{2}a_{23} & 0 & 0 \\ -\frac{1}{2}a_{11} & \frac{1}{2}a_{22} & -\frac{1}{2}a_{33} & 0 & -\frac{1}{2}a_{13} & 0 \\ -\frac{1}{2}a_{11} & -\frac{1}{2}a_{22} & \frac{1}{2}a_{33} & 0 & 0 & -\frac{1}{2}a_{12} \\ 0 & \frac{1}{2}a_{23} & \frac{1}{2}a_{23} & \frac{1}{4}(a_{33}+a_{22}) & \frac{1}{4}a_{12} & \frac{1}{4}a_{13} \\ \frac{1}{2}a_{13} & 0 & \frac{1}{2}a_{13} & \frac{1}{4}a_{12} & \frac{1}{4}(a_{11}+a_{33}) & \frac{1}{4}a_{23} \\ \frac{1}{2}a_{12} & \frac{1}{2}a_{12} & 0 & \frac{1}{4}a_{13} & \frac{1}{4}a_{23} & \frac{1}{4}(a_{11}+a_{22}) \end{bmatrix} \tag{41}$$

Inserting Eqs. (25—41) into Eq. (24) yields the equivalent matrix representation of Eq. (24)

$$\iiint_{\Omega}\delta\,\underline{\underline{\boldsymbol{\varepsilon}}}^{\mathrm{T}}\,\underline{\underline{\boldsymbol{\sigma}_{\text{eqsy}}}}\,\mathrm{d}\Omega-\iiint_{\Omega}\delta\,\underline{\underline{\boldsymbol{w}}}^{\mathrm{T}}\,\underline{\underline{\boldsymbol{\sigma}_{\text{eqan}}}}\,\mathrm{d}\Omega-\iiint_{\Omega}\delta\,\underline{\underline{\boldsymbol{E}}}^{\mathrm{T}}\,\underline{\underline{\boldsymbol{D}}}\mathrm{d}\Omega=\iint_{S_b}\delta\,\underline{\underline{\boldsymbol{u}}}^{\mathrm{T}}\,\underline{\underline{\boldsymbol{p}_b}}\mathrm{d}S+\iint_{S_b}\sigma_b\delta\phi\,\mathrm{d}S \tag{42}$$

Another equivalent expression of Eq. (42) can be acquired by introducing the generalized displacement vector, generalized stress tensor, generalized strain tensor and generalized surface force vector, where

$$\underline{\underline{\boldsymbol{u}_G}}=\begin{bmatrix}\underline{\underline{\boldsymbol{u}}}\\ \phi\end{bmatrix},\quad \underline{\underline{\boldsymbol{\sigma}_G}}=\begin{bmatrix}\underline{\underline{\boldsymbol{\sigma}_{\text{eqsy}}}}\\ \underline{\underline{\boldsymbol{\sigma}_{\text{eqan}}}}\\ \underline{\underline{\boldsymbol{D}}}\end{bmatrix}$$

$$\underline{\underline{\boldsymbol{\varepsilon}}}_G=\begin{bmatrix}\underline{\boldsymbol{\varepsilon}}\\-\underline{\boldsymbol{w}}\\-\underline{\underline{\boldsymbol{E}}}\end{bmatrix}=\begin{bmatrix}\underline{\underline{\boldsymbol{L}}}_{syu} & \mathbf{0}\\-\underline{\underline{\boldsymbol{L}}}_{anu} & \mathbf{0}\\\mathbf{0} & -\underline{\underline{\boldsymbol{L}}}_{\phi}\end{bmatrix}\begin{bmatrix}\underline{\boldsymbol{u}}\\\phi\end{bmatrix},\ \underline{\underline{\boldsymbol{p}}}_{Gb}=\begin{bmatrix}\underline{\boldsymbol{p}}_b\\\sigma_b\end{bmatrix} \tag{43}$$

Then Eqs. (28—30, 42) become

$$\underline{\underline{\boldsymbol{\sigma}}}_G=\begin{bmatrix}\underline{\underline{\boldsymbol{C}}}_{eq} & \mathbf{0} & -\underline{\underline{\boldsymbol{\beta}}}_{eq}\underline{\underline{\boldsymbol{B}}}_E\\\mathbf{0} & \mathbf{0} & -\underline{\underline{\boldsymbol{\lambda}}}_{eq}\underline{\underline{\boldsymbol{B}}}_E\\\mathbf{0} & \mathbf{0} & -\underline{\underline{\boldsymbol{\alpha}}}_{eq}\end{bmatrix}\underline{\underline{\boldsymbol{\varepsilon}}}_G=$$

$$\begin{bmatrix}\underline{\underline{\boldsymbol{C}}}_{eq} & \mathbf{0} & -\underline{\underline{\boldsymbol{\beta}}}_{eq}\underline{\underline{\boldsymbol{B}}}_E\\\mathbf{0} & \mathbf{0} & -\underline{\underline{\boldsymbol{\lambda}}}_{eq}\underline{\underline{\boldsymbol{B}}}_E\\\mathbf{0} & \mathbf{0} & -\underline{\underline{\boldsymbol{\sigma}}}_{eq}\end{bmatrix}\begin{bmatrix}\underline{\underline{\boldsymbol{L}}}_{syu} & \mathbf{0}\\-\underline{\underline{\boldsymbol{L}}}_{anu} & \mathbf{0}\\\mathbf{0} & -\underline{\underline{\boldsymbol{L}}}_{\phi}\end{bmatrix}\underline{\underline{\boldsymbol{u}}}_G \tag{44}$$

$$\iiint_{\Omega}\delta\,\underline{\underline{\boldsymbol{\varepsilon}}}_G^{\mathrm{T}}\,\underline{\underline{\boldsymbol{\sigma}}}_G\,\mathrm{d}\Omega=\iint_{S_b}\delta\,\underline{\underline{\boldsymbol{u}}}_G^{\mathrm{T}}\,\underline{\underline{\boldsymbol{p}}}_{Gb}\,\mathrm{d}S \tag{45}$$

The above work can degenerate to the uncoupled anisotropic case directly. One only needs to change Eq. (8) into

$$P_m=(\varepsilon_0\delta_{mk}-\rho_{\mathrm{mass}}\zeta_{mk})E_k \tag{46}$$

When degenerating to the coupled and uncoupled isotropic cases, one only has to set the coefficients in Eqs. (33, 34, 36) to be

$$\underline{\underline{\boldsymbol{\zeta}}}_{eq}=\begin{bmatrix}-\rho_{\mathrm{mass}}\zeta_{11} & 0 & 0\\0 & -\rho_{\mathrm{mass}}\zeta_{11} & 0\\0 & 0 & -\rho_{\mathrm{mass}}\zeta_{11}\end{bmatrix} \tag{47}$$

$$\underline{\underline{\boldsymbol{C}}}_{eq}=\begin{bmatrix}C_{eq11} & C_{eq12} & C_{eq12} & 0 & 0 & 0\\C_{eq12} & C_{eq11} & C_{eq12} & 0 & 0 & 0\\C_{eq12} & C_{eq12} & C_{eq11} & 0 & 0 & 0\\0 & 0 & 0 & C_{eq66} & 0 & 0\\0 & 0 & 0 & 0 & C_{eq66} & 0\\0 & 0 & 0 & 0 & 0 & C_{eq66}\end{bmatrix}$$

$$C_{eq66}=\frac{1}{2}(C_{eq11}-C_{eq12}) \tag{48}$$

$$\underline{\underline{\boldsymbol{\eta}}}_{eq}=\begin{bmatrix}\eta_{eq11} & \eta_{eq12} & \eta_{eq12} & 0 & 0 & 0\\\eta_{eq12} & \eta_{eq11} & \eta_{eq12} & 0 & 0 & 0\\\eta_{eq12} & \eta_{eq12} & \eta_{eq11} & 0 & 0 & 0\\0 & 0 & 0 & \eta_{eq66} & 0 & 0\\0 & 0 & 0 & 0 & \eta_{eq66} & 0\\0 & 0 & 0 & 0 & 0 & \eta_{eq66}\end{bmatrix}$$

$$\eta_{eq66}=\frac{1}{2}(\eta_{eq11}-\eta_{eq12}) \tag{49}$$

where

$$\zeta_{eq11}=-\rho_{\mathrm{mass}}\zeta_{11}=\varepsilon_0\chi_e \tag{50}$$

$$C_{eq12}=\lambda_{\mathrm{Lame}},\ C_{eq11}=2\mu_{\mathrm{Lame}}+\lambda_{\mathrm{Lame}} \tag{51}$$

$$\eta_{eq12}=-\alpha_{me},\ \eta_{eq11}=-(\alpha_{me}+\beta_{me}) \tag{52}$$

where ε_0 is the permittivity in vacuum, χ_e is the susceptibility of dielectric, λ_{Lame} and μ_{Lame} are Lame constants which can be expressed by Young's modulus E_{Yo} and Poisson ratio ν_{Po} as

$$\lambda_{\mathrm{Lame}}=\frac{E_{\mathrm{Yo}}\nu_{\mathrm{Po}}}{(1+\nu_{\mathrm{Po}})(1-2\nu_{\mathrm{Po}})},\ \mu_{\mathrm{Lame}}=\frac{E_{\mathrm{Yo}}}{2(1+\nu_{\mathrm{Po}})} \tag{53}$$

and α_{me} and β_{me} are the two independent electrostrictive coefficients representing the electro-mechanical coupling property.

5 Numerical Example Comparisons and Discussions

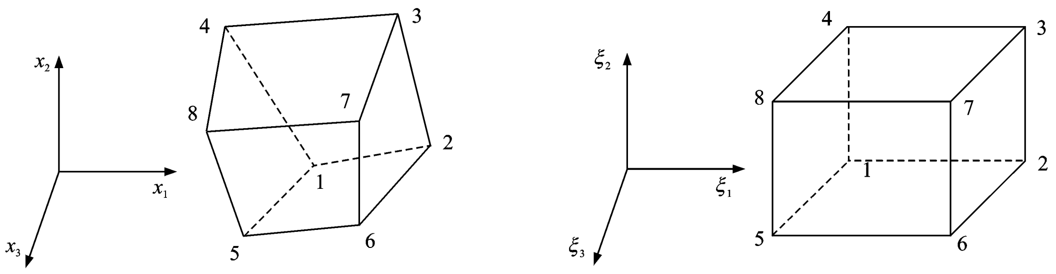
As shown in Fig. 2, a 3D hexahedral eight-node isoparametric element is introduced and defined in ABAQUS user subroutine UEL, where each node has degrees of freedom 1 (x_1-displacement), 2 (x_2-displacement), 3 (x_3-displacement), and 9 (electric potential). The dummy element method and ABAQUS user subroutine UVARM are used for post-processing.

Due to the lack of experiment data on anisotropic electrostrictive dielectric, we use one electrostrictive dielectric 0.9PMN:0.1PT (PMNT) for numerical examples[3,8]. The material constants are listed in Table 2.

In the following discussions, a finite isotropic electrostrictive plate containing an elliptical hole, where $l_1=100$ mm, $l_2=100$ mm, $l_3=1$ mm, $a=10$ mm, and $b=4$ mm, is used for illustration. A voltage of $\phi_{\mathrm{up}}-\phi_{\mathrm{down}}=5\times10^4$ V is applied which produces a $E_{2b}=0.5$ MV/m electric field along x_2-direction. And a voltage of $\phi_{\mathrm{up}}-\phi_{\mathrm{down}}=8\times10^4$ V would produce a 0.8 MV/m electric filed.

Figs. 3—6 plot coupled results of E_2, D_2, ε_{22} and σ_{22} around the elliptical on the un-deformed shape when $E_{2b}=0.8$ MV/m.

Figs. 7—10 and Table 3 compare the coupled and uncoupled FEM results of E_2, D_2, ε_{22} and σ_{22} along the elliptical hole where two different electric fields are applied. Besides, in order to validate our FEM programing, the uncoupled analytical results of generalized plane stress problem of an infinite isotropic electrostrictive plate



(a) Actual element (b) Transformed element

Fig. 2 Isoparametric transformation between two 3D hexahedral eight-node elements

Table 2 Material constants for PMNT

Material	E_{Yo}/GPa	ν_{Po}	ε_e	χ_e	α_{me}/(F·m^{-1})	β_{me}/(F·m^{-1})
0.9PMN:0.1PT	112	0.26	7 500ε_0	7 499	$-4.899\ 59\times10^{-6}$	$2.717\ 85\times10^{-5}$

where $\varepsilon_0=8.854\times10^{-12}$ F/m.

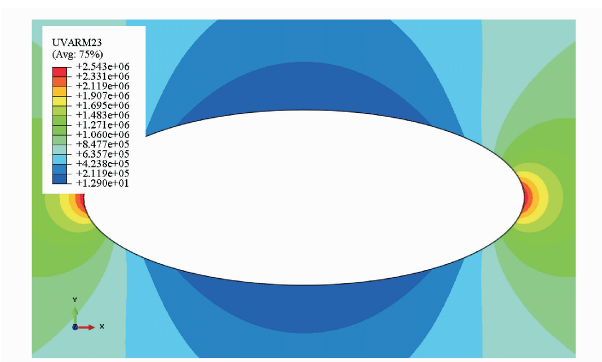


Fig. 3 Distribution of E_2 around elliptical hole in PMNT when $E_{2b}=0.8$ MV/m

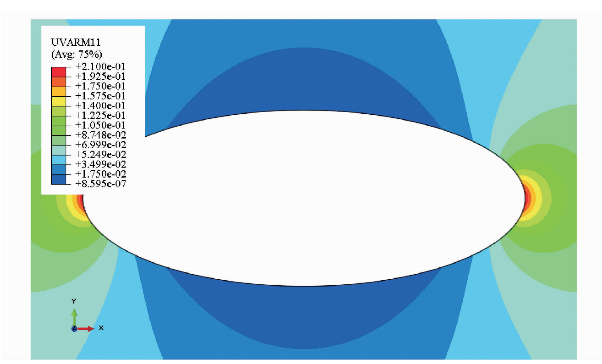


Fig. 4 Distribution of D_2 around elliptical hole in PMNT when $E_{2b}=0.8$ MV/m

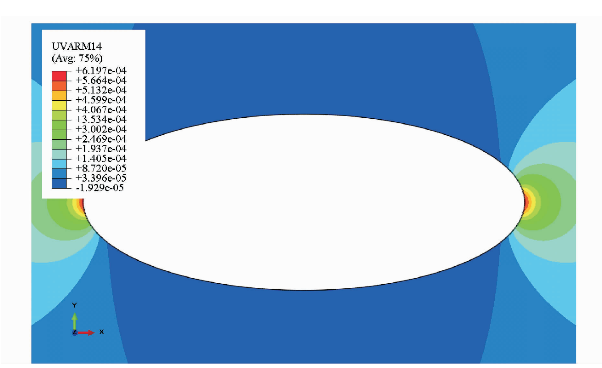


Fig. 5 Distribution of ε_{22} around elliptical hole in PMNT when $E_{2b}=0.8$ MV/m

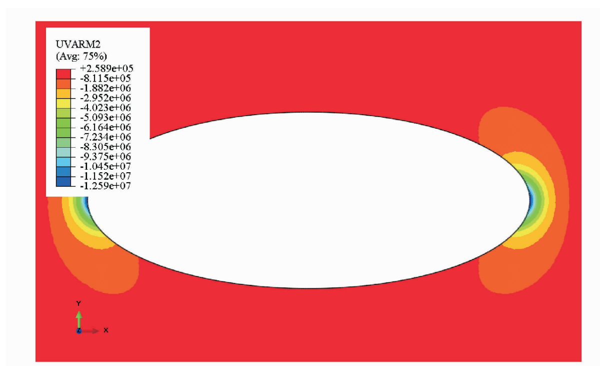


Fig. 6 Distribution of σ_{22} around elliptical hole in PMNT when $E_{2b}=0.8$ MV/m

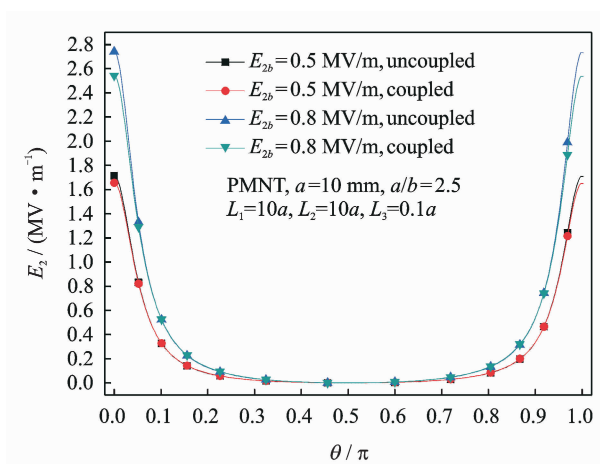


Fig. 7 Distribution of E_2 around elliptical hole in PMNT

with an elliptical hole is presented in Table 3 (see Ref. [9] for a detailed derivation).

According to Table 3, it is found that the uncoupled analytical results and uncoupled FEM results agree well, which supports our FEM programing to some extent.

Table 3 Comparisons of uncoupled analytical results, uncoupled FEM results and coupled FEM results

Parameter	E_{2b}=0.5 MV/m				E_{2b}=0.8 MV/m			
0.9PMN:0.1PT	I	II	III	(III/II)/%	I	II	III	(III/II)/%
$\|E_2\|_{max}/(MV\cdot m^{-1})$	1.750	1.713	1.656	96.67	2.800	2.741	2.543	92.78
$\|D_2\|_{max}(10^{-1}\ C\cdot m^{-2})$	1.16	1.138	1.212	106.5	1.859	1.820	2.100	115.4
$\|\varepsilon_{22}\|_{max}/10^{-4}$	2.843	2.667	2.604	97.64	7.279	7.086	6.197	87.45
$\|\sigma_{22}\|_{max}$/MPa	6.096	6.270	5.594	90.12	15.61	16.05	12.59	78.44

where I is the uncoupled analytical generalized plane stress solution of an infinite electrostrictive plate with an elliptical hole (a/b=2.5); II the uncoupled FEM solution of a finite electrostrictive plate with an elliptical hole (a=10 mm, b=4 mm, $L_1=L_2$=100 mm, L_3=1 mm); III the coupled FEM solution of a finite electrostrictive plate with an elliptical hole (a=10 mm, b=4 mm, $L_1=L_2$=100 mm, L_3=1 mm)

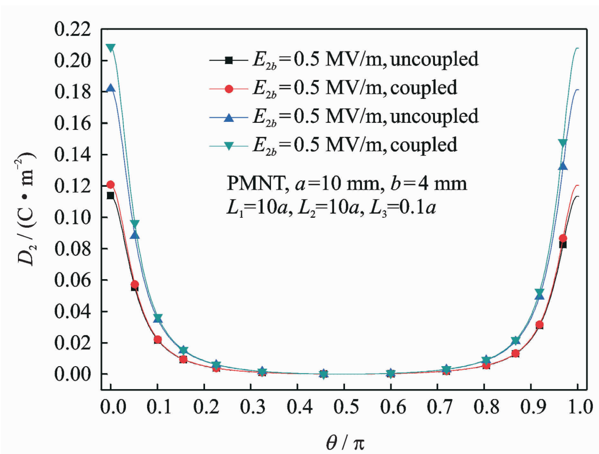


Fig. 8 Distribution of D_2 around elliptical hole in PMNT

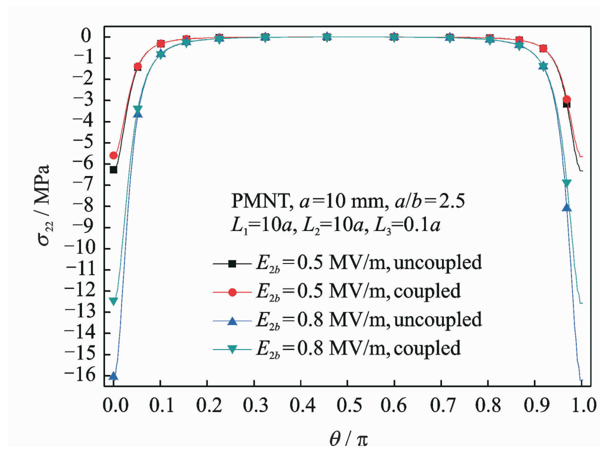


Fig. 10 Distribution of σ_{22} around elliptical hole in PMNT

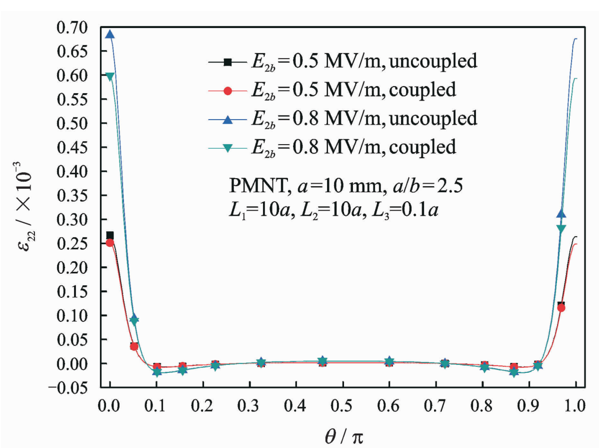


Fig. 9 Distribution of ε_{22} around elliptical hole in PMNT

From Fig. 7 and Table 3, it is found that (1) when E_{2b}=0.5 MV/m, the maximum value of E_2 based on the coupled method takes about 96.67% of that based on the uncoupled method; and (2) for E_{2b}=0.8 MV/m, the corresponding percentage is 92.78%. A decrease in E_2 does not mean the same deduction in D_2. To the contrary, Fig. 8 and Table 3 show the maximum value of D_2 based on the coupled method takes about 106.5% (for 0.5 MV/m) and 115.4% (for 0.8 MV/m) of those based on the uncoupled method, which displays an increasing trend in D_2. This is because the coupled method considers the effect of strain-stress field on polarization, and this kind of influence is strengthened by a higher applied electric field.

Fig. 9 and Table 3 show the maximum value of ε_{22} based on the coupled method takes about 97.64% (for 0.5 MV/m) and 87.45% (for 0.8 MV/m) of those based on the uncoupled method. While Fig. 10 and Table 3 show the maximum value of σ_{22} based on the coupled method takes about 90.12% (for 0.5 MV/m) and 78.44% (for 0.8 MV/m) of those based on the uncoupled method. This means the traditional uncoupled treatment may cause a large error, and a coupled analysis is especially needed in general.

It should be pointed out that when the electric potential or electric field is applied on the electrostrictive dielectric directly, the electric filed in the environment (vacuum or air) is considered to be zero, then the Maxwell stress contributed by the environment is zero. In this case, Fig. 10 shows that σ_{22} is negative at the end of the elliptical hole. If the electric is applied on the environment (vacuum or air), however, the Maxwell stress contributed by the environment will not be zero, and σ_{22} is positive at the end of the elliptical hole (see Ref. [9]). However, by taking our boundary condition, the work of Gao et al. [9] also results in a negative σ_{22}, which is consistent with our uncoupled FEM results.

Furthermore, numerical examples tell that when taking the contribution of strain-stress field to polarization into consideration, the strain-stress energy decreases, but the electric field energy increases.

6 Conclusions

This paper presents a different and detailed construction of coupled and uncoupled FEM analyses of anisotropic electrostrictive dielectric based on ABAQUS user subroutines UEL and UVARM. The developed technique is successfully applied to solve the electro-mechanical problem of an isotropic electrostrictive plate with an elliptical hole.

Although numerical examples use the isotropic materials, yet one only has to change the material constants in the FEM program when dealing with anisotropic materials.

Based on the numerical examples, it is found that (a) when the applied field becomes larger, the error between the coupled and uncoupled results increases; (b) the polarization induced by the strain-stress field may not be neglected at high electric fields. Furthermore, one may guess that a higher electrostrictive performance of the dielectric may lead to a higher error when applying the same electric field.

Generally speaking, it is necessary to evaluate polarization induced by the strain-stress field by using the third part of Eq. (8), that is, $\rho_{\text{mass}} \eta_{ijmn} \varepsilon_{ij} E_n$.

It shall be seen that, this method cannot be used to solve problems concerning thermal, piezoelectricity, etc.

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