Linear Codes with Two Weights or Three Weights from Two Types of Quadratic forms

Zhu Xiaoxing [1,2*], *Xu Dazhuan* [2]

1. Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, P. R. China;
2. College of Electronic and Information Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, P. R. China

(Received 8 January 2016; revised 30 October 2016; accepted 4 November 2016)

Abstract: Let $\mathbb{F}_q$ be a finite field with $q=p^m$ elements, where p is an odd prime and m is a positive integer. Here, let $D_0=\{(x_1,x_2)\in\mathbb{F}_q^2\backslash\{(0,0)\}: Tr(x_1^{p^{k_1}+1}+x_2^{p^{k_2}+1})=c\}$, where $c\in\mathbb{F}_q$, Tr is the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$ and $m/(m, k_1)$ is odd, $m/(m, k_2)$ is even. Define a p-ary linear code $C_D=c(a_1,a_2):(a_1,a_2)\in\mathbb{F}_q^2\}$, where $c(a_1, a_2)=(Tr(a_1x_1+a_2x_2))_{(x_1,x_2)\in D}$. At most three-weight distributions of two classes of linear codes are settled.

Key words: linear codes; weight distribution; Gauss sum; Weil sum

CLC number: 11T71;94B15 **Document code**: A **Article ID**: 1005-1120(2017)01-0062-10

0 Introduction

Let $\mathbb{F}_q$ be a finite field with $q=p^m$ elements throughout this paper, where p is an odd prime and m a positive integer, and let Tr be the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$. An $[n,k,d]$ p-ary linear code C is a k-dimensional subspace of $\mathbb{F}_q^n$ with the minimum Hamming distance d. Let A_i denote the number of codewords with Hamming weight i in code C of length n. The weight enumerator is defined by

$$1+A_1z+\cdots+A_nz^n$$

The sequence $(1,A_1,\cdots,A_n)$ is called the weight distribution of the code C. A code C is said to be a t-weight code if the number of nonzero A_i is equal to t. The weight distribution is an interesting topic and was investigated in Refs. [1-9]. Weight distribution of code can not only give the error correcting ability of the code, but also allow the computation of the error probability of error detection and correction.

For a set $D=\{d_1,d_2,\cdots,d_n\}\subseteq\mathbb{F}_q$, define a linear code of length n over $\mathbb{F}_p$ by

$$C_D=\{(Tr(xd_1),Tr(xd_2),\cdots,Tr(xd_n)):x\in\mathbb{F}_q\}$$

We call D the defining set of C_D. Many known linear codes could be produced by the selected defining sets. For more details of these codes, please refer to Refs. [1,3].

Here, assume that m,k_1,k_2 are the positive integers with $m/(m,k_1)$ being odd, $m/(m,k_2)$ being even. Then $f_1(x)=x^{p^{k_1}+1}$ is a planar function over $\mathbb{F}_q$[10]. Fixing $c\in\mathbb{F}_p$, we define

$$D=\{(x_1,x_2)\in\mathbb{F}_q^2\backslash\{(0,0)\}: Tr(x_1^{p^{k_1}+1}+x_2^{p^{k_2}+1})=c\}$$

$$C_D=\{c(a_1,a_2):(a_1,a_2)\in\mathbb{F}_q^2\}$$

where

$$c(a_1,a_2)=(Tr(a_1x_1+a_2x_2))_{(x_1,x_2)\in D}$$

In fact, we have some well-known results as follows. If $a_1=0$, then it is just the result in Ref. [11].

We will determine the weight distribution of the linear codes C_D in three cases: (1) $c=0$; (2) $c\in\mathbb{F}_p^{*2}$, (3) $c\in\mathbb{F}_p^*\backslash\mathbb{F}_p^{*2}$.

1 Preliminaries

Let $\mathbb{F}_q$ be a finite field with q elements,

* Corresponding author, E-mail address: zhuxiaoxing@nuaa.edu.cn.

How to cite this article: Zhu Xiaoxing, Xu Dazhuan. Linear codes with two weights or three weights from two types of quadratic forms[J]. Trans. Nanjing Univ. Aero. Astro., 2017,34(1):62-71.
http://dx.doi.org/10.16356/j.1005-1120.2017.01.062

where q is a power of a prime p. We define the additive character of $\mathbb{F}_q$ as follows

$$\chi:\mathbb{F}_q\to\mathbb{C}^*,\ x\to\zeta_p^{Tr(x)}$$

where ζ_p is a complex p-th primitive root of unity and Tr the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$. The orthogonal property of additive characters is given by

$$\sum_{x\in\mathbb{F}_q}\chi(ax)=\begin{cases}0 & a\in\mathbb{F}_q^*\\ q & a=0\end{cases}$$

Let $\lambda:\mathbb{F}_q^*\to\mathbb{C}^*$ be a multiplicative character of $\mathbb{F}_q^*$. The trivial character λ_0 defined by $\lambda_0(x)=1$ for all $x\in\mathbb{F}_q^*$. The orthogonal property of multiplicative characters is given by

$$\sum_{x\in\mathbb{F}_q^*}\lambda(x)=\begin{cases}q-1 & \lambda=\lambda_0\\ 0 & \text{Otherwise}\end{cases}$$

Let $\bar{\lambda}$ be the conjugate character of λ and it is defined by $\bar{\lambda}(x)=\overline{\lambda(x)}$. It is easy to obtain that $\lambda^{-1}=\bar{\lambda}$. The multiplicative group $\hat{\mathbb{F}}_q^*$ is isomorphic to $\mathbb{F}_q^*$. For $\mathbb{F}_q^*=\langle\alpha\rangle$, define a multiplicative character by $\psi(\alpha)=\zeta_{q-1}$, where ζ_{q-1} denotes the primitive root of complex unity. Then we have $\hat{\mathbb{F}}_q^*=\langle\psi\rangle$. Set $\eta=\psi^{\frac{q-1}{2}}$ the quadratic character of $\mathbb{F}_q$.

Define the Gauss sum over $\mathbb{F}_q$ by

$$G(\lambda)=\sum_{x\in\mathbb{F}_q^*}\lambda(x)\chi(x)$$

Let $(\frac{\cdot}{p})$ denote the Legendre symbol. The quadratic Gauss sums are known and given in the following.

Lemma 1.1[12] Suppose that $q=p^m$ and η is the quadratic multiplicative character of $\mathbb{F}_q$, where p is odd prime. Then

$$G(\eta)=(-1)^{m-1}\sqrt{(p^*)^m}=\begin{cases}(-1)^{m-1}\sqrt{q} & p\equiv1\pmod 4\\ (-1)^{m-1}(\sqrt{-1})^m\sqrt{q} & p\equiv 3\pmod 4\end{cases}$$

where $p^*=(-1)^{\frac{p-1}{2}}p$ is the discriminant of a prime p.

Lemma 1.2[12] Let χ be a nontrivial additive character of the $\mathbb{F}_q$ with q is odd, and let $f(x)=a_2x^2+a_1x+a_0\in\mathbb{F}_q[x]$ with $a_2\neq0$. Then

$$\sum_{c\in\mathbb{F}(q)}\chi(f(c))=\chi(a_0-a_1^2(4a_2)^{-1})\eta(a_2)G(\eta,\chi)$$

where η is the quadratic character of $\mathbb{F}_q$.

Let χ' be the canonical additive character of $\mathbb{F}_p$ such that $\chi(x)=\chi'(Tr(x))$ for $x\in\mathbb{F}_q$. Let η' be a quadratic character of $\mathbb{F}_p$, then $\eta(x)=\eta'(N_{q/p}(x))$ for $x\in\mathbb{F}_q^*$.

Lemma 1.3[13] Let $x\in\mathbb{F}_p^*$ and $q=p^m$, where p is odd prime.

If m is , $\eta(x)=1$.

If m is odd, $\eta(x)=\eta'(x)$.

Moreover, $G(\eta)=(-1)^{m-1}G(\eta')^m$, where $G(\eta)$ and $G(\eta')$ are the Gauss sums over $\mathbb{F}_q$ and $\mathbb{F}_p$, respectively.

We now give a brief introduction to the theory of quadratic forms over finite fields. Quadratic forms have been well studied and have been applied to sequence design[9,14] and coding theory[15].

Lemma 1.4 Let $d=\gcd(k,m)$. Then

$$(p^k+1,p^m-1)=\begin{cases}2 & m/d\text{ is odd}\\ p^d+1 & m/d\text{ is even}\end{cases}$$

In Refs. [16,17], Coulter gave the valuation of the following Weil sums

$$S_k(a,b)=\sum_{x\in\mathbb{F}_q}\chi(ax^{p^k+1}+bx)\qquad a,b\in\mathbb{F}_q$$

Lemma 1.5[16] Let m/d be odd. Then

$$S_k(a,0)=\eta(a)G(\eta)=\begin{cases}(-1)^{m-1}\sqrt{q}\eta(a) & p\equiv1\pmod 4\\ (-1)^{m-1}i^m\sqrt{q}\eta(a) & p\equiv3\pmod 4\end{cases}$$

Lemma 1.6[16] Let m/d be even with $m=2e$. Then

$$S_k(a,0)=\begin{cases}p^e & a^{(q-1)/(p^d+1)}\neq(-1)^{e/d}\text{ and } e/d \text{ is even}\\ -p^e & a^{(q-1)/(p^d+1)}\neq(-1)^{e/d}\text{ and } e/d \text{ is odd}\\ p^{e+d} & a^{(q-1)/(p^d+1)}=(-1)^{e/d}\text{ and } e/d \text{ is odd}\\ -p^{e+d} & a^{(q-1)/(p^d+1)}=(-1)^{e/d}\text{ and } e/d \text{ is even.}\end{cases}$$

Lemma 1.7[17] Let q be odd and suppose $f(X)=a^{p^k}X^{p^{2k}}+aX$ is a permutation polynomial over F_q. Let x_0 be the unique solution of the equation $f(X)=-b^{p^k}$. The evaluation of $S_k(a,b)$ partitions into the following two cases:

(1) If m/d is odd

$$S_k(a,b)=\eta(a)G(\eta)\bar{\chi}(ax_0^{p^k+1})=$$

$$\begin{cases}(-1)^{m-1}\sqrt{q}\eta(a)\bar{\chi}(ax_0^{p^k+1}) & p\equiv 1(\mathrm{mod}\ 4)\\(-1)^{m-1}\mathrm{i}^m\sqrt{q}\eta(a)\bar{\chi}(ax_0^{p^k+1}) & p\equiv 3(\mathrm{mod}\ 4)\end{cases}$$

(2) If m/d is even, $m=2e$, and $a^{\frac{q-1}{p^d+1}}\neq(-1)^{e/d}$

$$S_k(a,b)=(-1)^{e/d}p^e\bar{\chi}(ax_0^{p^k+1})$$

In fact, Lemma 1.4 is a revised version of the lemma in Ref. 4.

Lemma 1.8[17] Let q be odd and suppose $f(X)=a^{p^k}X^{p^{2k}}+aX$ is not a permutation polynomial over F_q. Then for $b\neq 0$, we have $S_k(a,b)=0$ unless the equation $f(X)=-b^{p^k}$ is solvable. If the equation is solvable, with some solution x_0 say, $S_k(a,b)=-(-1)^{e/d}p^{e+d}\bar{\chi}(ax_0^{p^k+1})$.

Lemma 1.9[11] Let $f(X)=X^{p^{2k}}+X$ and $S=\{b\in\mathbb{F}_q: f(X)=-b^{p^k}$ is solvable in $\mathbb{F}_q\}$, If m/d is even, $|S|=p^{2d}$.

2 Linear codes

Let $\mathbb{F}_q$ be the finite field with $q=p^m$ elements, where p is an odd prime and m an even positive integer with $m=2e$. Let Tr denote the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$. In this section, we always assume that e,d,k_1,k_2 are the positive integers with $m/\gcd(m,k_1)$ odd , $m/\gcd(m,k_2)$ even and $d=\gcd(m,k_2)$. Let $f_i(x)=x^{p^{k_i}+1}$, $x\in\mathbb{F}_q, i=1,2$.

2.1 The first case

Define

$$D_0=\{(x_1,x_2)\in\mathbb{F}_q^2\backslash\{(0,0)\}:\ \mathrm{Tr}(x_1^{p^{k_1}+1}+x_2^{p^{k_2}+1})=0\}\quad(1)$$

$$C_{D_0}=\{c(a_1,a_2):(a_1,a_2)\in\mathbb{F}_q^2\}$$

where $c(a_1,a_2)=(Tr(a_1x_1+a_2x_2))_{(x_1,x_2)\in D_0}$.

Lemma 2.1 Let $n_0=|D_0|$. Suppose that e/d is odd, then

$$n_0=\frac{q^2-p}{p}+\frac{p-1}{p}(-p^e)G(\eta)=\begin{cases}p^{2m-1}-1+(-1)^m(p-1)p^{m-1} & p\equiv 1\ (\mathrm{mod}\ 4)\\p^{2m-1}-1+(-1)^{m+\frac{m}{2}}(p-1)p^{m-1} & p\equiv 3\ (\mathrm{mod}\ 4)\end{cases}$$

Suppose that e/d is even, then

$$n_0=\frac{q^2-p}{p}+\frac{p-1}{p}(-p^{e+d})G(\eta)=\begin{cases}p^{2m-1}-1+(-1)^m(p-1)p^{m+d-1} & p\equiv 1\ (\mathrm{mod}\ 4)\\p^{2m-1}-1+(-1)^{m+\frac{m}{2}}(p-1)p^{m+d-1} & p\equiv 3\ (\mathrm{mod}\ 4)\end{cases}$$

Proof 1 By Lemmas 1.4, 1.5 and 1.6, we have that

$$n_0+1=\frac{1}{p}\sum_{y\in\mathbb{F}_p}\sum_{(x_1,x_2)\in\mathbb{F}_q^2}\chi(y(x_1^{p^{k_1}+1}+x_2^{p^{k_2}+1}))=$$

$$\frac{q^2}{p}+\frac{1}{p}\sum_{y\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\chi(yx_1^{p^{k_1}+1})\sum_{x_2\in\mathbb{F}_q}\chi(yx_2^{p^{k_2}+1})=$$

$$\begin{cases}\frac{q^n}{p}+\frac{1}{p}\sum_{y\in\mathbb{F}_p^*}G(\eta)\eta(y)(-p^e) & e/d\text{ is odd}\\\frac{q^n}{p}+\frac{1}{p}\sum_{y\in\mathbb{F}_p^*}G(\eta)\eta(y)(-p^{e+d}) & \frac{e}{d}\text{ is even}\end{cases}=$$

$$\begin{cases}\frac{q^2}{p}+\frac{p-1}{p}(-p^e)G(\eta) & e/d\text{ is odd}\\\frac{q^2}{p}+\frac{p-1}{p}(-p^{e+d})G(\eta) & e/d\text{ is even}\end{cases}$$

For each $y\in\mathbb{F}_p^*$, if m is even, $2\left|\frac{p^m-1}{p-1}\right.$ and $\eta(y)=1$;

For each $y\in\mathbb{F}_p^*$, we can know $y^{\frac{q-1}{p^d+1}}=1$. Then by Lemma 1.5, we can obtain the exact value of n_0.

Theorem 2.2 Let C_{D_0} be the linear code defined as Section 2.1.

If e/d is odd, C_{D_0} is a two-weight code with the Hamming weight distribution in Table 1.

If e/d is even, C_{D_0} is a three-weight code with the Hamming weight distribution in Table 2.

Table 1 e/d is odd

$p\equiv 1\ (\mathrm{mod}\ 4)$

Weight	Multiplicity
0	1
$p^{2m-1}-p^{2m-2}$	$\frac{q^2-p}{p}+(p-1)p^{m-1}$
$(p-1)(p^{2m-2}+(-1)^m p^{m-1})$	$\frac{p-1}{p}(p^{2m}+(-1)^{m-1}p^m)$

$p\equiv 3\ (\mathrm{mod}\ 4)$

Weight	Multiplicity
0	1
$p^{2m-1}-p^{2m-2}$	$\frac{q^2-p}{p}+(-1)^{\frac{3m}{2}}(p-1)p^{m-1}$
$(p-1)(p^{2m-2}+(-1)^{m+\frac{m}{2}}p^{m-1})$	$\frac{p-1}{p}(p^{2m}+(-1)^{(m-1)+\frac{m}{2}}p^m)$

Table 2 e/d is even

$p\equiv 1 \pmod 4$

Weight	Multiplicity
0	1
$(p-1)(p^{2m-2}+(-1)^m\cdot(p-1)p^{m+d-2})$	$p^{2m}-p^{m+2d}$
$p^{2m-1}-p^{2m-2}$	$(p-1)p^{m-d-1}+p^{m+2d-1}-1$
$(p-1)(p^{2m-2}+(-1)^m p^{m+d-1})$	$(p-1)(p^{m+2d-1}-p^{m-d-1})$

$p\equiv 3 \pmod 4$

Weight	Multiplicity
0	1
$(p-1)(p^{2m-2}+(-1)^{m+\frac{m}{2}}(p-1)p^{m+d-2})$	$p^{2m}-p^{m+2d}$
$p^{2m-1}-p^{2m-2}$	$(p-1)p^{m-d-1}+p^{m+2d-1}-1$
$(p-1)(p^{2m-2}+(-1)^{m+\frac{m}{2}}p^{m+d-1})$	$(p-1)(p^{m+2d-1}-p^{m-d-1})$

Proof 2 Firstly, we determine the weight distribution of the code C_{D_0}. Define the following parameter

$$N_a=|\{(x_1,x_2)\in\mathbb{F}_q^2\setminus\{(0,0)\}:\ Tr(x_1^{p^{k_1}+1}+x_2^{p^{k_2}+1})=0,\ Tr(a_1x_1+a_2x_2)=0\}|$$

where $a=(a_1,a_2)\in\mathbb{F}_q^2$. By definition and the basic facts of additive characters, for each $a=(a_1,a_2)\in\mathbb{F}_q^n\setminus\{(0,0)\}$, we have

$$N_a=\frac{1}{p^2}\sum_{(x_1,x_2)\in\mathbb{F}_q^2}\sum_{y\in\mathbb{F}_p}\chi(y(x_1^{p^{k_1}+1}+x_2^{p^{k_2}+1}))\cdot\sum_{z\in\mathbb{F}_p}\chi(z(a_1x_1+a_2x_2))-1=$$

$$\frac{q^2-p^2}{p^2}+\frac{1}{p^2}\sum_{(x_1,x_2)\in\mathbb{F}_q^2}\sum_{y\in\mathbb{F}_p^*}\chi(y(x_1^{p^{k_1}+1}+x_2^{p^{k_2}+1}))+\frac{1}{p^2}\sum_{(x_1,x_2)\in\mathbb{F}_q^2}\sum_{z\in\mathbb{F}_p^*}\chi(z(a_1x_1+a_2x_2))+\frac{1}{p^2}\sum_{(x_1,x_2)\in\mathbb{F}_q^2}\sum_{y,z\in\mathbb{F}_p^*}\chi(y(x_1^{p^{k_1}+1}+x_2^{p^{k_2}+1})+z(a_1x_1+a_2x_2))=:$$

$$\frac{q^2-p^2}{p^2}+\Omega_1+\Omega_2+\Omega_3$$

By Lemmas 1.5 and 1.6, we have

$$\Omega_1=\frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\chi(yx_1^{p^{k_1}+1})\sum_{x_2\in\mathbb{F}_q}\chi(yx_2^{p^{k_2}+1})=$$

$$\begin{cases}\dfrac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}G(\eta)\eta(y)(-p^e) & e/d \text{ is odd}\\ \dfrac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}G(\eta)\eta(y)(-p^{e+d}) & e/d \text{ is even}\end{cases}=$$

$$\begin{cases}\dfrac{p-1}{p^2}(-p^e)G(\eta) & \dfrac{e}{d} \text{ is odd}\\ \dfrac{p-1}{p^2}(-p^{e+d})G(\eta) & \dfrac{e}{d} \text{ is even}\end{cases}$$

By $a=(a_1,a_2)\in\mathbb{F}_q^2\setminus\{(0,0)\}$, we have

$$\Omega_2=\frac{1}{p^2}\sum_{z\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\chi(za_1x_1)\cdot\sum_{x_2\in\mathbb{F}_q}\chi(za_2x_2)=0$$

To compute N_a, it is sufficient to determine the value of the exponential sum

$$\Omega_3=\frac{1}{p^2}\sum_{y,z\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\chi(yx_1^{p^{k_1}+1}+za_1x_1)\cdot\sum_{x_2\in\mathbb{F}_q}\chi(yx_n^{p^{k_2}+1}+za_2x_2)$$

For $y\in\mathbb{F}_p^*$, we set $d=\gcd(k,m)$, when m/d is even, $m=2e$, e/d is even, we know the polynomial $f_1(x)=y^{p^{k_1}}x^{p^{2k_1}}+yx$ is not a permutation polynomial over $\mathbb{F}_q$; when m/d is odd or m/d is even, $m=2e$, e/d is odd, the polynomial $f_1(x)=y^{p^{k_1}}x^{p^{2k_1}}+yx=y(x^{p^{2k_1}}+x)$ must be a permutation polynomial over $\mathbb{F}_q$.

In fact, suppose that there is $0\neq b\in\mathbb{F}_q$ such that $f_1(b)=0$. Then $b^{p^{2k_1}-1}=-1$. Let α be a primitive element of $\mathbb{F}_q^*$ and $b=\alpha^t$, then

$$t(p^{2k_1}-1)\equiv\frac{p^m-1}{2}\pmod{p^m-1}\qquad(2)$$

Let $d_1=\gcd(m,k_1)$, then $\gcd(2k_1,m)=d_1$ by m/d_1 odd. Hence $\gcd(p^{2k_1}-1,p^m-1)=(p^{d_1}-1)$ and $(p^{d_1}-1)\dfrac{p^m-1}{2}$, so Eq. (2) is contradictory.

Since $f_i(x)=y(x^{p^{2k_i}}+x)$ is a permutation poly-nomial over $\mathbb{F}_q$, for each $a_i\in\mathbb{F}_q$ there is the unique solution $b_i\in\mathbb{F}_q$ of the equation $x_i^{p^{2k_i}}+x_i+a_i^{p^{k_i}}=0$. In fact, there is a one-to-one correspondence between $a_i\in\mathbb{F}_q$ and $b_i\in\mathbb{F}_q$, and $a_i=0$ corresponds to $b_i=0$.

Hence there is the unique solution $wb_i\in\mathbb{F}_q$ of the equation $y(x_i^{p^{2k_i}}+x_i+wa_i^{p^{k_i}})=0$, where $w=\dfrac{z}{y}\in\mathbb{F}_p^*$.

To compute the value of Ω_3, we divide into two cases.

The first case: e/d is odd. By Lemma 1.7, we have

$$\Omega_3=\frac{1}{p^2}\sum_{y,w\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\chi(yx_1^{p^{k_1}+1}+ywa_1x_1)\cdot\sum_{x_2\in\mathbb{F}_q}\chi(yx_2^{p^{k_2}+1}+ywa_2x_2)=$$
$$\frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\eta(y)G(\eta)(-1)^{\frac{e}{d}}p^e\cdot\sum_{w\in\mathbb{F}_p^*}\chi'\Big(y\sum_{i=1}^{2}(wb_i)^{p^{k_i}+1}\Big)=$$
$$\frac{-p^e}{p^2}\sum_{y\in\mathbb{F}_p^*}\eta(y)G(\eta)\cdot\sum_{w\in\mathbb{F}_p^*}\chi'\Big(yw^2Tr\Big(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\Big)\Big)$$

Set

$$\Gamma_0=\{(b_1,b_2)\in\mathbb{F}_q^2,\{(0,0)\}\mid Tr\Big(\sum_{i=1}^{2}b_i^{p^{k_i}+1}\Big)=0\}$$

$$\Gamma_0'=\Big\{(b_1,b_2)\in\mathbb{F}_q^2\,|\,Tr\Big(\sum_{i=1}^{2}b_i^{p^{k_i}+1}\Big)\neq 0\Big\}$$

Since m is even, then we have $\eta(y)=1$ for $y\in\mathbb{F}_p^*$.

If $(b_1,b_2)\in\Gamma_0$

$$\Omega_3=\frac{(-p^e)(p-1)^2}{p^2}\cdot G(\eta)$$

$$N_a=\frac{q^2-p^2}{p^2}+\frac{(-p^e)(p-1)}{p^2}G(\eta)+\frac{(-p^e)(p-1)^2}{p^2}G(\eta)=\frac{q^2-p^2}{p^2}+\frac{(-p^e)(p-1)}{p}G(\eta)$$

Hence, by Lemma 2.1, the weight of C_a is

$$n_0-N_a=\frac{q^2-p}{p}+\frac{p-1}{p}(-p^e)G(\eta)-\frac{q^2-p^2}{p^2}-\frac{(-p^e)(p-1)}{p}G(\eta)=p^{2m-1}-p^{2m-2}$$

If $(b_1,b_2)\in\Gamma_0'$

$$\Omega_3=\frac{-p^e}{p^2}G(\eta)\sum_{y,w\in\mathbb{F}_p^*}\chi'\Big(yw^2Tr\Big(\sum_{i=1}^{n}b_i^{p^{k_i}+1}\Big)\Big)=\frac{(p-1)p^e}{p^2}G(\eta)$$

$$N_a=\frac{q^2-p^2}{p^2}$$

By Lemma 2.1, the weight of C_a is

$$n_0-N_a=\frac{q^2-p}{p}+\frac{p-1}{p}(-p^e)G(\eta)-\frac{q^2-p^2}{p^2}=\begin{cases}(p-1)(p^{2m-2}+(-1)^m p^{m-1}) & p\equiv 1\pmod 4\\(p-1)(p^{2m-2}+(-1)^{m+\frac{m}{2}}p^{m-1}) & p\equiv 3\pmod 4\end{cases}$$

The second case: e/d is even. By Lemmas 1.7, 1.8 and 1.9, if $a_2\in\mathbb{F}_q\backslash S$

$$\Omega_3=\frac{1}{p^2}\sum_{y,w\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\chi(yx_1^{p^{k_1}+1}+ywa_1x_1)\sum_{x_2\in\mathbb{F}_q}\chi(yx_2^{p^{k_2}+1}+ywa_2x_2)=\frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}0=0$$

$$N_a=\frac{q^2-p^2}{p^2}+\frac{(-p^{e+d})(p-1)}{p^2}G(\eta)$$

By Lemma 2.1, the weight of C_a is

$$n_0-N_a=\frac{q^2-p}{p}+\frac{p-1}{p}(-p^{e+d})G(\eta)-\frac{q^2-p^2}{p^2}-\frac{(-p^{e+d})(p-1)}{p^2}G(\eta)=$$
$$p^{2m-1}-p^{2m-2}+\frac{(-p^{e+d})(p-1)^2}{p^2}G(\eta)=$$
$$\begin{cases}(p-1)(p^{2m-2}+(-1)^m(p-1)p^{m+d-2}) & p\equiv 1\pmod 4\\(p-1)(p^{2m-2}+(-1)^{m+\frac{m}{2}}(p-1)p^{m+d-2}) & p\equiv 3\pmod 4\end{cases}$$

If $a_2\in S$

$$\Omega_3=\frac{1}{p^2}\sum_{y,w\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q}\chi(yx_1^{p^{k_1}+1}+ywa_1x_1)\cdot\sum_{x_2\in\mathbb{F}_q}\chi(yx_2^{p^{k_2}+1}+ywa_2x_2)=$$
$$\frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\eta(y)G(\eta)(-1)^{\frac{e}{d}+1}p^{e+d}\cdot\sum_{w\in\mathbb{F}_p^*}\chi'\Big(y\sum_{i=1}^{2}(wb_i)^{p^{k_i}+1}\Big)=$$
$$\frac{-p^{e+d}}{p^2}\sum_{y\in\mathbb{F}_p^*}\eta(y)G(\eta)\sum_{w\in\mathbb{F}_p^*}\chi'\Big(yw^2Tr\Big(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\Big)\Big)$$

If $(b_1,b_2)\in\Gamma_0$

$$\Omega_3=\frac{(-p^{e+d})(p-1)^2}{p^2}G(\eta)$$

$$N_a=\frac{q^2-p^2}{p^2}+\frac{(-p^{e+d})(p-1)}{p^2}G(\eta)+\frac{(-p^{e+d})(p-1)^2}{p^2}G(\eta)=\frac{q^2-p^2}{p^2}+\frac{(-p^{e+d})(p-1)}{p}G(\eta)$$

By Lemma 2.1, the weight of C_a is

$$n_0 - N_a = \frac{q^2-p}{p} + \frac{p-1}{p}(-p^{e+d})G(\eta) - \frac{q^2-p^2}{p^2} - \frac{(-p^{e+d})(p-1)}{p}G(\eta) = p^{2m-1} - p^{2m-2}$$

If $(b_1, b_2) \in \Gamma_0'$

$$\Omega_3 = \frac{-p^{e+d}}{p^2}G(\eta)\sum_{y,w\in\mathbb{F}_p^*}\chi'(yw^2 Tr(\sum_{i=1}^{n} b_i^{p^{k_i}+1})) = \frac{(p-1)p^{e+d}}{p^2}G(\eta)$$

$$N_a = \frac{q^2-p^2}{p^2}$$

By Lemma 2.1, the weight of C_a is

$$n_0 - N_a = \frac{q^2-p}{p} + \frac{p-1}{p}(-p^{e+d})G(\eta) - \frac{q^2-p^2}{p^2} = \begin{cases}(p-1)(p^{2m-2}+(-1)^m p^{m+d-1}) \\ \quad p\equiv 1 \pmod 4 \\ (p-1)(p^{2m-2}+(-1)^{m+\frac{m}{2}} p^{m+d-1}) \\ \quad p\equiv 3 \pmod 4\end{cases}$$

Secondly, we determine the frequency of each nonzero weight of C_{D_0}. It is sufficient to consider the values of $|\Gamma_0|, |\Gamma'_0|$.

By Lemma 2.1, it is clear that

$$|\Gamma_0| = n_0 = \begin{cases}\frac{q^2-p}{p} + \frac{p-1}{p}(-p^e)G(\eta) \\ \quad e/d \text{ is odd} \\ \frac{q^2-p}{p} + \frac{p-1}{p}(-p^{e+d})G(\eta) \\ \quad e/d \text{ is even}\end{cases}$$

Since $|\Gamma_0| < q^n$. Without loss of generality, suppose that $\Gamma_0' \neq \varnothing$. If e/d is odd, then for each $c \in \mathbb{F}_p^*$, there are $(x_1, x_2) \in \mathbb{F}_q^2$ such that $Tr(x_1^{p^{k_1}+1} + x_2^{p^{k_2}+1}) = c \in \mathbb{F}_p^*$. Hence

$$|\Gamma'_0| = \frac{p-1}{p}\sum_{y\in\mathbb{F}_p}\sum_{(x_1,x_2)\in\mathbb{F}_q^2}\chi'(yTr(x_1^{p^{k_1}+1} + x_2^{p^{k_2}+1}) - cy) = \frac{(p-1)q^2}{p} + \frac{p-1}{p}\sum_{y\in\mathbb{F}_p^*}\chi'(-cy)$$

$$\sum_{x_1\in\mathbb{F}_q}\chi(yx_1^{p^{k_1}+1})\sum_{x_2\in\mathbb{F}_q}\chi(yx_2^{p^{k_2}+1}) = \frac{(p-1)q^2}{p} + \frac{p-1}{p}\sum_{y\in\mathbb{F}_p^*}G(\eta)\eta(y)(-p^e)\chi'(-cy) =$$

$$\begin{cases}\frac{p-1}{p}(p^{2m}+(-1)^{m-1}p^m) & p\equiv 1 \pmod 4 \\ \frac{p-1}{p}(p^{2m}+(-1)^{(m-1)+\frac{m}{2}}p^m) & p\equiv 3 \pmod 4\end{cases}$$

If e/d is even, we have

$$wt(c_b) \in \{p^{2m-1} - p^{2m-2} + \frac{(-p^{e+d})(p-1)^2}{p^2}G(\eta), p^{2m-1} - p^{2m-2}, (p-1)(p^{2m-2} + (-1)p^{e+d-1}G(\eta))\}$$

By Lemma 1.9, $A_{b_1} = p^{2m} - p^{m+2d}$. By the first two Pless Power Monments[18], the frequency A_{b_i} of b_i satisfies

$$\begin{cases}A_{b_1} + A_{b_2} + A_{b_3} = p^{2m} - 1 \\ b_1 A_{b_1} + b_2 A_{b_2} + b_3 A_{b_3} = p^{2m-1}(p-1)n\end{cases}$$

where $n = \frac{q^2}{p} + \frac{p-1}{p}(-p^{e+d})G(\eta) - 1$. Hence, we obtain Tables 1,2.

2.2 The second case

Fix $c \in \mathbb{F}_p^{*2}$ and define

$$D_1 = \{(x_1, x_2) \in \mathbb{F}_q^2 : Tr(x_1^{p^{k_1}+1} + x_2^{p^{k_2}+1}) = c\}$$

$$C_{D_1} = \{c(a_1, a_2) : (a_1, a_2) \in \mathbb{F}_q^2\} \tag{3}$$

where $c(a_1, a_2) = (Tr(a_1x_1 + a_2x_2))_{(x_1,x_2)\in D_1}$.

Since C_{D_1} is a linear code over $\mathbb{F}_p$, it is independent of the choice of c. For convenience, we take $c=1$.

By Lemma 1.1 and the computation of Γ_0' as above, we can get the result.

Lemma 2.3 Let $n_1 = |D_1|$. Suppose that e/d is odd, then

$$n_1 = \begin{cases}\frac{1}{p}(p^{2m} + (-1)^{m-1}p^m) \\ \quad p\equiv 1 \pmod 4 \\ \frac{1}{p}(p^{2m} + (-1)^{(m-1)+\frac{m}{2}}p^m) \\ \quad p\equiv 3 \pmod 4\end{cases}$$

Suppose that e/d is even, then

$$n_1 = \begin{cases}\frac{1}{p}(p^{2m} + (-1)^{m-1}p^{m+d}) \\ \quad p\equiv 1 \pmod 4 \\ \frac{1}{p}(p^{2m} + (-1)^{(m-1)+\frac{m}{2}}p^{m+d}) \\ \quad p\equiv 3 \pmod 4\end{cases}$$

Theorem 2.4 Let C_{D_1} be the linear code defined as Eq. (3).

If e/d is odd, C_{D_1} is a two-weight code with the Hamming weight distribution in Table 3.

If e/d is even, C_{D_1} is a three-weight code with the Hamming weight distribution in Table 4.

Table 3 e/d is odd

$p\equiv 1 \pmod 4$

Weight	Multiplicity
0	1
$(p-1)p^{2m-2}$	$\frac{p^{2m}(p+1)}{2p}+\frac{1}{2}(p-1)p^{m-1}-1$
$(p-1)p^{2m-2}+(-1)^{(m-1)}2p^{m-1}$	$\frac{p-1}{2p}(p^{2m}+(-1)^{m-1}p^{m})$

$p\equiv 3 \pmod 4$

Weight	Multiplicity
0	1
$(p-1)p^{2m-2}+(-1)^{(m-1+\frac{m}{2})}2p^{m-1}$	$\frac{p^{2m}(p+1)}{2p}+\frac{(-1)^{\frac{3m}{2}}}{2}\cdot(p-1)p^{m-1}-1$
$(p-1)p^{2m-2}$	$\frac{p-1}{2p}(p^{2m}+(-1)^{(m-1)+\frac{m}{2}}p^{m})$

Table 4 e/d is even

$p\equiv 1 \pmod 4$

Weight	Multiplicity
0	1
$(p-1)(p^{2m-2}+(-1)^{m-1}p^{m+d-2})$	$p^{2m}-p^{m+2d}$
$(p-1)p^{2m-2}$	$\frac{p-1}{2}(p^{m+2d-1}-p^{m-d-1})$
$(p-1)p^{2m-2}+(-1)^{(m-1)}2p^{m+d-1}$	$p^{m+2d}-\frac{p-1}{2}\cdot(p^{m+2d-1}-p^{m-d-1})-1$

$p\equiv 3 \pmod 4$

Weight	Multiplicity
0	1
$(p-1)(p^{2m-2}+(-1)^{m-1}p^{m+d-2})$	$p^{2m}-p^{m+2d}$
$(p-1)p^{2m-2}+(-1)^{(m-1)}2p^{m+d-1}$	$\frac{p-1}{2}(p^{m+2d-1}-p^{m-d-1})$
$(p-1)p^{2m-2}$	$p^{m+2d}-\frac{p-1}{2}\cdot(p^{m+2d-1}-p^{m-d-1})-1$

Proof 3 Firstly, we determine the weight distribution of the code C_{D_1}. Fix $c=1\in\mathbb{F}_p^*$. Define the following parameter

$$N'_a=|\{(x_1,x_2)\in\mathbb{F}_q^2:Tr(x_1^{p^{k_1}+1}+x_2^{p^{k_2}+1})=1,Tr(a_1x_1+a_2x_2)=0\}|$$

where $a=(a_1,a_2)\in\mathbb{F}_q^2\backslash\{(0,0)\}$, we have

$$N'_a=\frac{1}{p^2}\sum_{(x_1,x_2)\in\mathbb{F}_q^2,y,z\in\mathbb{F}_p}\chi'(y(Tr(x_1^{p^{k_1}+1}+x_2^{p^{k_2}+1})-1))\chi(z(a_1x_1+a_2x_2))=$$

$$\frac{q^2}{p^2}+\frac{1}{p^2}\sum_{(x_1,x_2)\in\mathbb{F}_q^2}\sum_{y\in\mathbb{F}_p^*}\chi'(y(Tr(x_1^{p^{k_1}+1}+x_2^{p^{k_2}+1})-1))+\frac{1}{p^2}\sum_{(x_1,x_2)\in\mathbb{F}_q^2}\sum_{z\in\mathbb{F}_p^*}\chi(z(a_1x_1+a_2x_2))+\frac{1}{p^2}$$

$$\sum_{(x_1,x_2)\in\mathbb{F}_q^2}\sum_{y,z\in\mathbb{F}_p^*}\chi'(y(Tr(x_1^{p^{k_1}+1}+x_2^{p^{k_2}+1})-1))\cdot\chi(z(a_1x_1+a_2x_2))=:\frac{q^2}{p^2}+\Omega'_1+\Omega'_2+\Omega'_3$$

We have

$$\Omega'_1=\frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\chi'(-y)\sum_{x_1\in\mathbb{F}_q}\chi(yx_1^{p^{k_1}+1})\sum_{x_2\in\mathbb{F}_q}\chi(yx_2^{p^{k_2}+1})=\begin{cases}-\frac{1}{p^2}G(\eta)(-p^e) & e/d\text{ is odd}\\ -\frac{1}{p^2}G(\eta)(-p^{e+d}) & e/d\text{ is even}\end{cases}$$

By $(a_1,a_2)\in\mathbb{F}_p^2\backslash\{(0,0)\}$, we have that

$$\Omega'_2=0.$$

Similarly, we have

$$\Omega'_3=\frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\eta(y)G(\eta)(-p^e)\cdot\sum_{w\in\mathbb{F}_p^*}\chi'\Big(y\Big(w^2Tr\Big(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\Big)-1\Big)\Big)$$

where (b_1,b_2) is one-to-one correspondent to (a_1,a_2).

To compute the value of Ω'_3, we divide it into two cases.

The first case: e/d is odd.

If $(b_1,b_2)\in\Gamma_0$

$$\Omega'_3=\frac{p-1}{p^2}G(\eta)(-p^e)\sum_{y\in\mathbb{F}_p^*}\chi'(-y)=(p-1)p^{e-2}G(\eta)$$

$$N'_a=\frac{q^2}{p^2}+p^{e-2}G(\eta)+(p-1)p^{e-2}G(\eta)=p^{2m-2}+p^{e-1}G(\eta)$$

$$n_1-N'_a=(p-1)p^{2m-2}$$

If $(b_1,b_2)\in\Gamma'_0$, $w^2\ Tr(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1})\neq 0$ for any $w\in\mathbb{F}_p^*$. By Lemma 1.2

$$\Omega'_3=\frac{-p^e}{p^2}G(\eta)\Big(\sum_{y,w\in\mathbb{F}_p^*}\eta(y)\chi'(y(w^2)\cdot Tr\Big(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\Big)-1)\Big)\Big)=$$

$$\frac{-p^e}{p^2}G(\eta)\sum_{y\in\mathbb{F}_q^*}\Big(\sum_{w\in\mathbb{F}_p}\chi'(y(w^2Tr\Big(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\Big)-1)\Big)-\chi'(-y)\Big)=$$

$$\frac{-p^e}{p^2}G(\eta)-\frac{p^e}{p^2}G(\eta)\sum_{y\in\mathbb{F}_p^*}\eta'(y$$

$$Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)\chi'(-y)G(\eta')=$$

$$\frac{-p^e}{p^2}G(\eta)-\frac{-p^e}{p^2}G(\eta)G(\eta')^2\eta'\cdot$$

$$\left(-Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)\right)$$

$$N'_a=\frac{q^2}{p^2}+p^{e-2}G(\eta)-p^{e-2}G(\eta)+p^{e-2}\cdot$$

$$G(\eta)G(\eta')^2\eta'\left(-Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)\right)$$

$$n_1-N'_a=\frac{(p-1)q^2}{p^2}+p^{e-1}G(\eta)-$$

$$p^{e-2}G(\eta)G(\eta')^2\eta'\left(-Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)\right)=$$

$$\begin{cases}(p-1)p^{2m-2}-p^{m-1}+\\ \quad p^{m-1}\eta'(Tr(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1})) \quad p\equiv 1 \pmod 4\\ (p-1)p^{2m-2}+(-1)^{\frac{m}{2}-1}p^{m-1}+\\ \quad (-1)^{\frac{m}{2}-1}p^{m-1}\eta'\left(Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)\right)\\ \quad p\equiv 3 \pmod 4\end{cases}$$

Since $|\Gamma_0|<q^n$. Without loss of generality, suppose that $\Gamma_2\neq\varnothing$. For some $c\in\mathbb{F}_q^{*2}$, there are $(x_1,x_2)\in\mathbb{F}_q^2$ such that $Tr(x_1^{p_{k_1}+1}+(x_2^{p_{k_1}+1})=c\in\mathbb{F}_p^{*2}$. By the property of the trace function, the values are presented averagely from $\mathbb{F}_p^{*2}$.

Hence, set

$$\Gamma_1=\{(b_1,b_2)\in\mathbb{F}_q^{*2}\mid Tr\left(\sum_{i=1}^{2}b_i^{p^{k_i}+1}\right)\in\mathbb{F}_p^{*2}$$

$$\Gamma_2=\{(b_1,b_2)\in\mathbb{F}_q^{*2}\mid Tr\left(\sum_{i=1}^{2}b_i^{p^{k_i}+1}\right)\in\mathbb{F}_p^{*}\setminus\mathbb{F}_p^{*2}$$

$$|\Gamma_1|=\frac{p-1}{2p}\sum_{y\in\mathbb{F}_p}\sum_{(x_1,x_2)\in\mathbb{F}_q^2}\chi'(yTr(x_1^{p^{k_1}+1}+$$

$$x_2^{p^{k_2}+1})-cy)=\frac{q^2(p-1)}{2p}+$$

$$\frac{p-1}{2p}\sum_{y\in\mathbb{F}_p^*}\chi'(-cy)\sum_{x_1\in\mathbb{F}_q}\chi(yx_1^{p^{k_1}+1})$$

$$\sum_{x_2\in\mathbb{F}_q}\chi(yx_2^{p^{k_2}+1})=\frac{(p-1)q^2}{2p}-$$

$$\frac{p-1}{2p}G(\eta)\eta(y)p^e\sum_{y\in\mathbb{F}_p^*}\chi'(-cy)=\frac{(p-1)q^2}{2p}+$$

$$\frac{p-1}{2p}p^eG(\eta)=|\Gamma_2|$$

The second case: e/d is even.

If $a_2\in\mathbb{F}_q\setminus S$

$$\Omega'_3=\frac{1}{p^2}\sum_{y,w\in\mathbb{F}_p^*}\chi'(-y)\sum_{x_1\in\mathbb{F}_q}\chi(yx_1^{p^{k_1}+1}+ywa_1x_1)\cdot$$

$$\sum_{x_2\in\mathbb{F}_q}\chi(yx_2^{p^{k_2}+1}+ywa_2x_2)=\frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\chi'(-y)0=0$$

$$N'_a=\frac{q^2}{p^2}+p^{e+d-2}G(\eta)$$

By Lemma 2.1, the weight of C_a is

$$n_1-N'_a=\frac{q^2}{p}+p^{e+d-1}G(\eta)-\frac{q^2}{p^2}-p^{e+d-2}G(\eta)=$$

$$p^{2m-1}-p^{2m-2}+(p-1)p^{e+d-2}G(\eta)=$$

$$\begin{cases}(p-1)(p^{2m-2}+(-1)^{m-1}p^{m+d-2})\\ \quad p\equiv 1 \pmod 4\\ (p-1)(p^{2m-2}+(-1)^{m+\frac{m-1}{2}}p^{m+d-2})\\ \quad p\equiv 3 \pmod 4\end{cases}$$

If $a_2\in S$

$$\Omega'_3=\frac{1}{p^2}\sum_{y\in\mathbb{F}_p^*}\eta(y)G(\eta)(-1)^{\frac{e}{d}+1}p^{e+d}\cdot$$

$$\sum_{w\in\mathbb{F}_p^*}\chi'\left(y\left(w^2Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)-1\right)\right)$$

If $(b_1,b_2)\in\Gamma_0$

$$\Omega'_3=\frac{p-1}{p^2}G(\eta)(-p^{e+d})\cdot$$

$$\sum_{y\in\mathbb{F}_p^*}\chi'(-y)=(p-1)p^{e+d-2}G(\eta),$$

$$N'_a=\frac{q^2}{p^2}+p^{e+d-1}G(\eta)$$

$$n_1-N'_a=(p-1)p^{2m-2}$$

If $(b_1,b_2)\in\Gamma'_0$ $w^2Tr(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1})\neq 0$ for any $w\in\mathbb{F}_p^*$. By Lemma 1.2

$$\Omega'_3=\frac{-p^{e+d}}{p^2}G(\eta)\Big(\sum_{y,w\in\mathbb{F}_p^*,}\eta(y)\chi'(y(w^2\cdot$$

$$Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)-1)\Big)\Big)=$$

$$\frac{-p^{e+d}}{p^2}G(\eta)\sum_{y\in\mathbb{F}_q^*}\Big(\sum_{w\in\mathbb{F}_p}\chi'(y(w^2$$

$$Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)-1)\Big)-\chi'(-y)\Big)=$$

$$\frac{-p^{e+d}}{p^2}G(\eta)-\frac{-p^{e+d}}{p^2}G(\eta)G(\eta')^2\eta'$$

$$\left(-Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)\right)$$

Hence

$$N'_a=\frac{q^2}{p^2}+p^{e+d-2}G(\eta)-p^{e+d-2}G(\eta)+$$

$$p^{e+d-2}G(\eta)G(\eta')^2\eta'\left(-Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)\right)$$

$$n_1-N'_a=\frac{(p-1)q^2}{p^2}+p^{e+d-1}G(\eta)-p^{e+d-2}\cdot$$

$$G(\eta)G(\eta')^2\eta'\left(-Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)\right)=$$
$$\begin{cases}(p-1)p^{2m-2}-p^{m+d-1}+p^{m+d-1}\eta' \\ \quad\left(Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)\right) \quad p\equiv 1\ (\mathrm{mod}\ 4) \\ (p-1)p^{2m-2}-p^{m+d-1}-p^{m+d-1}\eta' \\ \quad\left(Tr\left(\sum_{i=1}^{2}(b_i)^{p^{k_i}+1}\right)\right) \quad p\equiv 3(\mathrm{mod}\ 4)\end{cases}$$

Suppose that

$$b_1=(p-1)p^{2m-2}+(p-1)p^{e+d-2}G(\eta)$$
$$b_2=(p-1)p^{2m-2}$$
$$b_3=(p-1)p^{2m-2}+2p^{e+d-1}G(\eta)$$

By Lemma 1. 9, $A_{b_1}=p^{2m}-p^{m+2d}$. By the first two Pless Power Moments[18], the frequency A_{b_i} of b_i satisfies the following equations

$$\begin{cases}A_{b_1}+A_{b_2}+A_{b_3}=p^{2m}-1 \\ b_1A_{b_1}+b_2A_{b_2}+b_3A_{b_3}=p^{2m-1}(p-1)n\end{cases}$$

where $n=\frac{1}{p}(p^{2m}-p^{m+d})$. Hence, we obtain Tables 3,4.

In Eq. (3), suppose that $c\in\mathbb{F}_p^*\setminus\mathbb{F}_p^{*2}$. Similarly, we can get the weight distribution of the linear code. The case can be omitted.

3 Conclusions

There is a recent survey on three-weight codes[19-23]. We remark that the third class of binary codes is new. We did not find the parameters of the binary three-weight codes of this paper in these literatures.

Linear codes can be used to construct secret sharing schemes [24]. Let $w_{\min}$ and $w_{\max}$ denote the minimum and maximum nonzero Hamming weights of a linear code C. To obtain secret sharing schemes with interesting access structures, we would like to construct linear codes with the property that

$$w_{\min}/w_{\max}>\frac{p-1}{p}$$

We remark that the linear codes in this paper can be employed in secret sharing schemes using the framework in Ref. [24].

References:

[1] CARLET C, DING C. Highly nonlinear mappings [J]. Journal of Complexity, 2004, 20(2): 205-244.

[2] CALDERBANK R, KANTOR W M. The geometry of two-weight codes [J]. Bulletin of the London Mathematical Society, 1986, 18: 97-122.

[3] DING C. Linear codes from some 2-designs [J]. IEEE Transaction of Information Theory, 2015, 61 (6): 2065-2075.

[4] YANG S, YAO Z A. Complete weight enumerators of a family of three-weight linear codes[J]. Designs Codes and Cryptography, 2015: 1-12.

[5] YANG S, YAO Z A, ZHAO C A. A class of three-weight linear codes and their complete weight enumerators[J]. Cryptography Communications, 2016, 9(1):1-17.

[6] HENG Z L, YUE Q. Two classes of two-weight linear codes [J]. Finite Fields and Their Applications, 2016, 38: 72-92.

[7] ZHOU Z, DING C. Seven classes of three-weight cyclic codes[J]. IEEE Transaction of Information Theory, 2013, 61(10): 4120-4126.

[8] ZENG X, HU L, JIANG W, et al. The weight distribution of a class of p-ary cyclic codes [J]. Finite Fields and Their Applications, 2010, 16: 56-73.

[9] ZENG X, LIU J Q, HU L. Generalized Kasami sequences: The large set [J]. IEEE Transaction of Information Theory, 2007, 53(7): 2587-2598.

[10] COULTER R S, MATTHEWS R W. Planar functions and planes of Lenz-Barlotti class II [J]. Design Codes and Cryptography, 1997, 10: 167-184.

[11] LI F, WANG Q, LIN D. Complete weight enumerators of a class of three-weight linear codes[J]. Journal of Applied Mathematics & Computing, 2016:1-15.

[12] LIDL R, NIEDERREITER H. Finite fields [M]. [S. l.]: Addison-Wesley Publishing Inc., 1983.

[13] BERNDT B, EVANS R, WILLIAMS K. Gauss and Jacobi Sums [M]. New York: John Wiley and Sons Company, 1997.

[14] LUO J Q, TANG Y, WANG H. Cyclic codes and sequences: The generalized Kasami case [J]. IEEE Transaction of Information Theory, 2010, 56(5): 2130-2142.

[15] ZHOU Z, DING C. A class of three-weight cyclic codes [J]. Finite Fields and Their Applications, 2014, 25: 79-93.

[16] COULTER R S. Explicit evaluations of some Weil sums [J]. Acta Arithmetrica, 1998, 83(3): 241-251.

[17] COULTER R S. Further evaluations of Weil sums

[J]. Acta Arithmetrica, 1998, 86(3): 217-226.

[18] HUFFMAN W C, PLESS V. Fundamentals of error-correcting codes [M]. Cambridge: Cambridge University Press, 2003.

[19] DING K L, DING C S. A class of two-weight and three weight codes and their applicationsin secret sharing [J]. IEEE Transaction of Information Theory, 2015, 61(11): 5835-5842.

[20] DING K L, DING C S. Binary linear codes with three weights [J]. IEEE Communication Letters, 2014, 18(11): 1879-1882.

[21] LI C J, YUE Q, LI F W. Weight distributions of cyclic codes with respect to pairwise coprime order elements[J]. Finite Fields and Their Applications, 2014, 28: 94-114.

[22] LI C J, YUE Q, LI F W. Hamming weights of the duals of cyclic codes with two zeros [J]. IEEE Transaction of Information Theory, 2014, 60(7): 3895-3902.

[23] ZHU X, YANG F. A class of linear codes with two weights or three weights from some planar functions [J]. Journal of Applied Mathematics & Computing, 2016:1-18.

[24] YUAN J, DING C. Secret sharing schemes from three classes of linear codes [J]. IEEE Transaction of Information Theory, 2006, 52(1): 206-212.

Dr. **Zhu Xiaoxing** received his B. S. and M. S. degrees in Maths. department from Yangzhou University in 1999 and 2002, respectively. He joined in Nanjing university of Aeronautics and Astronautics in June, 2002. His research focuses on algebraic coding theory.

Prof. **Xu Dazhuan** received his M. S. and Ph. D degrees in electrical engineering in 1986 and 2001, respectively, from Nanjing University of Aeronautics and Astronautics, Nanjing, China. His current research interests include wireless communication, network and communications signal processing.

(Executive Editor: Zhang Tong)