Perturbation Theory of Fractional Lagrangian System and Fractional Birkhoffian System

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Abstract: Perturbation to symmetry and adiabatic invariants are studied for the fractional Lagrangian system and the fractional Birkhoffian system in the sense of Riemann-Liouville derivatives. Firstly, the fractional Euler-Lagrange equation, the fractional Birkhoff equations as well as the fractional conservation laws for the two systems are listed. Secondly, the definition of adiabatic invariant for fractional mechanical system is given, then perturbation to symmetry and adiabatic invariants are established for the fractional Lagrangian system and the fractional Birkhoffian system under the special and general infinitesimal transformations, respectively. Finally, two examples are devoted to illustrate the results.

Key words: perturbation theory; fractional conservation law; Riemann-Liouville derivative; fractional Euler-Lagrange equation; fractional Birkhoff equation

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0 Introduction

In 1917, adiabatic invariant was first proposed by Burgers[1]. A certain physical quantity is called adiabatic invariant of a system if it varies more slowly than the parameters which change very slowly. In fact, the parameter changing very slowly can be expressed as the action of small disturbance. Under the action of small disturbance, the original symmetry and conserved quantity may change. At the same time, because perturbation to symmetry and adiabatic invariant concern the integrability of the equations of motion of mechanical systems, they were studied by many scientists, and many important results were obtained[2-9]. However, almost all of those results about adiabatic invariant referred to only integer order derivatives of the variables. Therefore, there is still much to do on the aspect of the non-integer order derivatives of the variables. Hence, in this paper, we intend to study perturbation to symmetry and adiabatic invariant in terms of fractional calculus.

Fractional calculus has been studied for more than 300 years by many famous mathematicians, and many significant results about fractional calculus have been obtained[10-17]. Besides, based on the fractional calculus, Riewe[18-19] investigated the version of the Euler-Lagrange equations for the problem of the calculus of variations with fractional derivatives under the conservative and non-conservative cases respectively. Since then, many further studies on fractional problems can be found[20-38]. For example, in 2002, Agrawal[20] proved a formulation for the variational problem in the sense of Riemann-Liouville derivatives. Then Baleanu and Avkar[26] used those Euler-Lagrange equations to study the problem with Lagrangian which is linear on the velocities. Frederico and Torres[27] used the notion of the Euler-Lagrange fractional extremal[20] to prove a

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Noether-type theorem. Using the similar method adopted in Ref. [27], Zhou[39] studied the fractional Pfaff-Birkhoff principle in terms of Riemann-Liouville derivatives, and obtained the fractional Birkhoff equations, the corresponding transversality conditions and the fractional-conserved quantities. Based on the results of Refs. [27,39], we intend to study the adiabatic invariant of the fractional calculus of variations.

1 Preliminaries

In this section, some relevant knowledge would be recalled.

Definition 1[14] Let f be a continuous and integrable function in the interval $[t_1, t_2]$, for all $t \in [t_1, t_2]$, the left Riemann-Liouville fractional derivative ${}_{t_1}D_t^{\alpha} f(t)$ of order α, and the right Riemann-Liouville fractional derivative ${}_tD_{t_2}^{\beta} f(t)$ of order β, are defined as follows

$$ {}_{t_1}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \times \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_{t_1}^{t} (t-\theta)^{n-\alpha-1} f(\theta)\,\mathrm{d}\theta \tag{1} $$

$$ {}_tD_{t_2}^{\beta} f(t) = \frac{1}{\Gamma(m-\beta)} \times \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_{t}^{t_2} (\theta-t)^{m-\beta-1} f(\theta)\,\mathrm{d}\theta \tag{2} $$

where $\Gamma(\cdot)$ is the Euler Gamma function, α, β are the orders of the derivatives satisfying $n-1 \leqslant \alpha < n, m-1 \leqslant \beta < m, m, n \in \mathbf{N}$. If α, β are integers, those derivatives are defined in the usual sense, that is

$$ {}_{t_1}D_t^{\alpha} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} f(t) $$

$$ {}_tD_{t_2}^{\beta} f(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\beta} f(t) \tag{3} $$

In this paper, we assume that $0 < \alpha < 1, 0 < \beta < 1$.

In Ref. [20], Agrawal considered the functional

$$ I[q(\cdot)] = \int_a^b L(t, q(t), {}_aD_t^{\alpha} q(t), {}_tD_b^{\beta} q(t))\,\mathrm{d}t \tag{4} $$

where $q(a) = q_a, q(b) = q_b$ and the Lagrangian L: $[a,b] \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is a C^2 function with respect to all its arguments. And he got the following fractional Euler-Lagrange equation in terms of Riemann-Liouville derivatives

$$ \partial_2 L(t, q, {}_aD_t^{\alpha} q, {}_tD_b^{\beta} q) + {}_tD_b^{\alpha} \partial_3 L(t, q, {}_aD_t^{\alpha} q, {}_tD_b^{\beta} q) + {}_aD_t^{\beta} \partial_4 L(t, q, {}_aD_t^{\alpha} q, {}_tD_b^{\beta} q) = 0 \tag{5} $$

In Ref. [39], Zhou and Zhang studied the extremum for the following functional

$$ S(a^{\mu}(\cdot)) = \int_{t_1}^{t_2} (R_{\nu}^{\alpha}\, {}_{t_1}D_t^{\alpha} a^{\nu} + R_{\nu}^{\beta}\, {}_tD_{t_2}^{\beta} a^{\nu} - B)\,\mathrm{d}t \qquad \mu, \nu = 1, 2, \cdots, 2n \tag{6} $$

where $R_{\nu}^{\alpha} = R_{\nu}^{\alpha}(t, a^{\mu}), R_{\nu}^{\beta} = R_{\nu}^{\beta}(t, a^{\mu})$ are the Birkhoff's functions, $B = B(t, a^{\mu})$ is the Birkhoffian, and they are both C^2 functions with respect to all their arguments. And they obtained the following fractional Birkhoff equations

$$ \frac{\partial R_{\nu}^{\alpha}}{\partial a^{\mu}}\, {}_{t_1}D_t^{\alpha} a^{\nu} + \frac{\partial R_{\nu}^{\beta}}{\partial a^{\mu}}\, {}_tD_{t_2}^{\beta} a^{\nu} - \frac{\partial B}{\partial a^{\mu}} + {}_tD_{t_2}^{\alpha} R_{\mu}^{\alpha} + {}_{t_1}D_t^{\beta} R_{\mu}^{\beta} = 0 \qquad \mu = 1, 2, \cdots, 2n \tag{7} $$

Definition 2[27] Given two functions $f, g \in C^1[a,b]$, we introduce the following notation

$$ D_t^{\gamma}(f, g) = -g\, {}_tD_b^{\gamma} f + f\, {}_aD_t^{\gamma} g \tag{8} $$

where $t \in [a,b]$, and $\gamma \in \mathbf{R}_0^+$.

The linearity of the operators ${}_aD_t^{\gamma}$ and ${}_tD_b^{\gamma}$ implies the linearity of the operator D_t^{γ}.

If $\gamma = 1$, the operator D_t^{γ} reduces to

$$ D_t^1(f, g) = -g\, {}_tD_b^1 f + f\, {}_aD_t^1 g = g\dot{f} + f\dot{g} = \frac{\mathrm{d}}{\mathrm{d}t}(fg) \tag{9} $$

2 Fractional Adiabatic Invariants

In this section, we study adiabatic invariants under the general and special infinitesimal transformations for the fractional Lagrangian system and the fractional Birkhoffian system.

2.1 Adiabatic invariants for the fractional Lagrangian system

Firstly, let's consider only the infinitesimal transformation for q

$$ \bar{t} = t, \bar{q}(t) = q(t) + \varepsilon\zeta(t, q) + o(\varepsilon) \tag{10} $$

where ζ is called the infinitesimal generator.

Theorem 1[27] Under the infinitesimal transformation (10), if the condition

$$ \partial_2 L(t, q, {}_aD_t^{\alpha} q, {}_tD_b^{\beta} q) \cdot \zeta + \partial_3 L(t, q, {}_aD_t^{\alpha} q, {}_tD_b^{\beta} q) \cdot {}_aD_t^{\alpha}\zeta + \partial_4 L(t, q, {}_aD_t^{\alpha} q, {}_tD_b^{\beta} q) \cdot {}_tD_b^{\beta}\zeta = 0 \tag{11} $$

holds, then

$$c_f^L = [\partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) - \partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q)] \cdot \zeta \quad (12)$$

is a fractional-conserved quantity.

Theorem 2[27] Under the infinitesimal transformations

$$\bar{t} = t + \varepsilon\tau(t,q) + o(\varepsilon)$$
$$\bar{q}(t) = q(t) + \varepsilon\zeta(t,q) + o(\varepsilon) \quad (13)$$

if functional (4) is invariant, i. e.

$$\int_{t_a}^{t_b} L(t,q(t),{}_aD_t^\alpha q(t),{}_tD_b^\beta q(t))\,dt = \int_{\bar{t}(t_a)}^{\bar{t}(t_b)} L(\bar{t},\bar{q}(\bar{t}),{}_{\bar{a}}D_{\bar{t}}^\alpha \bar{q}(\bar{t}),{}_{\bar{t}}D_{\bar{b}}^\beta \bar{q}(\bar{t}))\,d\bar{t} \quad (14)$$

for any subinterval $[t_a,t_b] \subseteq [a,b]$

$$c_{Lf} = [\partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) - \partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q)] \cdot \zeta + [L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) - \alpha\partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) \cdot {}_aD_t^\alpha q - \beta\partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) \cdot {}_tD_b^\beta q] \cdot \tau \quad (15)$$

is a fractional-conserved quantity.

Definition 3 If

$$\sum_{j=0}^{z}\sum_{i=1}^{m}\varepsilon^j [D_t^\omega(c_i^1,c_i^2)]_j \quad \omega \in \{\alpha,\beta\}$$

is in direct proportion to ε^{z+1}

$$I_z = \sum_{j=0}^{z}\sum_{i=1}^{m}\varepsilon^j (c_i^1 \cdot c_i^2)_j$$

is called a z-th order adiabatic invariant of a fractional order dynamical system.

For the fractional Lagrangian system (Eq. (5)), if ζ_0 satisfies Eq. (11), the following exact invariant exists

$$I_0^L = [\partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) - \partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q)] \cdot \zeta_0 \quad (16)$$

Similarly, if τ_0,ζ_0 satisfy Eq. (14), the exact invariant exists as follows

$$I_{L0} = [\partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) - \partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q)] \cdot \zeta_0 + [L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) - \alpha\partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) \cdot {}_aD_t^\alpha q] - \beta\partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) \cdot {}_tD_b^\beta q \cdot \tau_0 \quad (17)$$

Suppose the fractional Lagrangian system (Eq. (5)) is disturbed by small quantity εQ, then we can get the disturbed fractional Euler-Lagrange equation

$$\partial_2 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) + {}_tD_b^\alpha \partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) + {}_aD_t^\beta \partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) = -\varepsilon Q \quad (18)$$

Under the action of small force of perturbation εQ, the invariant of the system may vary. Suppose that the disturbed infinitesimal generator ζ can be expressed as

$$\zeta = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \cdots \quad (19)$$

we have Theorem 3 as follow.

Theorem 3 For the disturbed fractional Lagrangian system (Eq. (18)), if the infinitesimal generators $\zeta_j\,(j=0,1,2,\cdots)$ satisfy

$$\partial_2 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) \cdot \zeta_j + \partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) \cdot {}_aD_t^\alpha \zeta_j + \partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) \cdot {}_tD_b^\beta \zeta_j + Q\zeta_{j-1} = 0 \quad (20)$$

the disturbed fractional Lagrangian system has a z-th order adiabatic invariant

$$I_z^L = \sum_{j=0}^{z}\varepsilon^j [\partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) - \partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q)] \cdot \zeta_j \quad (21)$$

where we set $\zeta_{j-1}=0$, when $j=0$.

Proof From the disturbed fractional Euler-Lagrange equation and the condition, we have

$$\sum_{j=0}^{z}\varepsilon^j [D_t^\alpha(\partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q),\zeta_j) - D_t^\beta(\zeta_j,\partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q))] = \sum_{j=0}^{z}\varepsilon^j [-\zeta_j \cdot {}_tD_b^\alpha \partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) - \partial_2 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) \cdot \zeta_j - Q\zeta_{j-1} - \zeta_j \cdot {}_aD_t^\beta \partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q)] = \sum_{j=0}^{z}\varepsilon^j(-Q\zeta_{j-1} + \varepsilon Q\zeta_j) = \varepsilon^{z+1}Q\zeta_z$$

Hence, the proof is completed.

Theorem 4 Under the infinitesimal transformations

$$\bar{t} = t + \varepsilon\tau(t,q) + o(\varepsilon)$$
$$\bar{q}(t) = q(t) + \varepsilon\zeta(t,q) + o(\varepsilon) \quad (22)$$

where

$$\tau = \tau_0 + \varepsilon\tau_1 + \varepsilon^2\tau_2 + \cdots$$
$$\zeta = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \cdots \quad (23)$$

the disturbed fractional Lagrangian system (Eq. (18)) has a z-th order adiabatic invariant

$$I_{Lz} = \sum_{j=0}^{z}\varepsilon^j \{[\partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q) - \partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q)] \cdot \zeta_j +$$

$$[L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q)-$$
$$\alpha\partial_3 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q)\,{}_aD_t^\alpha q-$$
$$\beta\partial_4 L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q)\,{}_tD_b^\beta q]\cdot\tau_j\}\quad(24)$$

Proof In order to consider t as a dependent variable, we use a Lipschitzian one-to-one transformation

$$[a,b]\ni t\to\sigma f(\lambda)\in[\sigma_a,\sigma_b]\qquad(25)$$

which satisfies $t'_\sigma=f(\lambda)=1$ when $\lambda=0$, $t(\sigma_a)=a$, $t(\sigma_b)=b$.

From the definitions of the right Riemann-Liouville fractional derivative and the left Riemann-Liouville fractional derivative, we have

$${}_{\sigma_a}D_{t(\sigma)}^\alpha q(t(\sigma))=(t'_\sigma)^{-\alpha}\;{}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q(\sigma)$$
$${}_{t(\sigma)}D_{\sigma_b}^\beta q(t(\sigma))=(t'_\sigma)^{-\beta}\;{}_\sigma D^\beta_{\frac{b}{(t'_\sigma)^2}}q(\sigma)\qquad(26)$$

Hence

$$\bar I[t(\cdot),q(t(\cdot))]=\int_{\sigma_a}^{\sigma_b}L\,(t(\sigma),q(t(\sigma)),$$
$${}_{\sigma_a}D_{t(\sigma)}^\alpha q(t(\sigma)),{}_{t(\sigma)}D_{\sigma_b}^\beta q(t(\sigma)))\,t'_\sigma\mathrm{d}\sigma=$$
$$\int_{\sigma_a}^{\sigma_b}L\,(t(\sigma),q(t(\sigma)),(t'_\sigma)^{-\alpha}\times$$
$${}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q(\sigma),(t'_\sigma)^{-\beta}\,{}_\sigma D^\beta_{\frac{b}{(t'_\sigma)^2}}q(\sigma))\,t'_\sigma\mathrm{d}\sigma\doteq$$
$$\int_{\sigma_a}^{\sigma_b}\bar L\,(t(\sigma),q(t(\sigma)),t'_\sigma,{}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q(\sigma),$$
$${}_\sigma D^\beta_{\frac{b}{(t'_\sigma)^2}}q(\sigma))\,\mathrm{d}\sigma=$$
$$\int_a^b L\,(t,q(t),{}_aD_t^\alpha q(t),{}_tD_b^\beta q(t))\,\mathrm{d}t=$$
$$I[q(\cdot)]$$

From Theorem 3, we can obtain

$$I_{Lz}\Big(t(\sigma),q(t(\sigma)),t'_\sigma,$$
$${}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q(\sigma),{}_\sigma D_{\frac{b}{(t'_\sigma)^2}}{}^\beta q(\sigma)\Big)=$$
$$\sum_{j=0}^{z}\varepsilon^j\left[(\partial_4\bar L-\partial_5\bar L)\cdot\zeta_j+\frac{\partial\bar L}{\partial t'_\sigma}\tau_j\right]\qquad(27)$$

If $\lambda=0$, we can get

$${}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q(\sigma)={}_aD_t^\alpha q(t)$$
$${}_\sigma D^\beta_{\frac{b}{(t'_\sigma)^2}}q(\sigma)={}_tD_b^\beta q(t)\qquad(28)$$
$$\partial_4\bar L-\partial_5\bar L=\partial_3 L-\partial_4 L\qquad(29)$$
$$\frac{\partial\bar L}{\partial t'_\sigma}=-\alpha\partial_3 L\cdot{}_aD_t^\alpha q-\beta\partial_4 L\cdot{}_tD_b^\beta q+L\quad(30)$$

Therefore, when $\lambda=0$, we have

$$I_{Lz}(t,q(t),{}_aD_t^\alpha q(t),{}_tD_b^\beta q(t))=$$
$$\sum_{j=0}^{z}\varepsilon^j[(\partial_3 L-\partial_4 L)\cdot\zeta_j+$$
$$(-\alpha\partial_3 L\cdot{}_aD_t^\alpha q-\beta\partial_4 L\cdot{}_tD_b^\beta q+L)\tau_j]$$

The proof is completed.

2.2 Adiabatic invariants for the fractional Birkhoffian system

We consider only the infinitesimal transformations for a^μ

$$\bar a^\nu(t)=a^\nu(t)+\varepsilon\xi_\nu(t,a^\mu)+o(\varepsilon)$$
$$\mu,\nu=1,2,\cdots,2n;\bar t=t\qquad(31)$$

where $\xi_\nu(\nu=1,2,\cdots,2n)$ are called the infinitesimal generators.

Theorem 5[39] Under the infinitesimal transformations (Eq. (31)), if

$$\frac{\partial R_\nu^\alpha}{\partial a^\mu}\xi_\mu\,{}_{t_1}D_t^\alpha a^\nu+\frac{\partial R_\nu^\beta}{\partial a^\mu}\xi_\mu\,{}_tD_{t_2}^\beta a^\nu+$$
$$R_\nu^\alpha\,{}_{t_1}D_t^\alpha\xi_\nu+R_\nu^\beta\,{}_tD_{t_2}^\beta\xi_\nu-\frac{\partial B}{\partial a^\mu}\xi_\mu=0\qquad(32)$$

we have

$$C_f^B(t,a^\mu,{}_{t_1}D_t^\alpha a^\mu,{}_tD_{t_2}^\beta a^\mu)=$$
$$[R_\nu^\alpha(t,a^\mu)-R_\nu^\beta(t,a^\mu)]\,\xi_\nu(t,a^\mu)\qquad(33)$$

is a fractional-conserved quantity.

Therefore, for the fractional Birkhoffian system (Eq. (7)), if ξ_ν^0 satisfies Eq. (32), exact invariant exists as follows

$$I_0^B=[R_\nu^\alpha(t,a^\mu)-R_\nu^\beta(t,a^\mu)]\,\xi_\nu^0\qquad(34)$$

Theorem 6[39] Under the infinitesimal transformations

$$\bar t=t+\varepsilon\xi_0(t,a^\mu)+o(\varepsilon)$$
$$\bar a^\nu(t)=a^\nu(t)+\varepsilon\xi_\nu(t,a^\mu)+o(\varepsilon)$$
$$\mu,\nu=1,2,\cdots,2n\qquad(35)$$

if functional (6) is invariant, i. e.

$$\int_{T_1}^{T_2}(R_\nu^\alpha(t,a^\mu)\,{}_{t_1}D_t^\alpha a^\nu+R_\nu^\beta(t,a^\mu)\,{}_tD_{t_2}^\beta a^\nu-$$
$$B(t,a^\mu))\,\mathrm{d}t=$$
$$\int_{\bar T_1}^{\bar T_2}(R_\nu^\alpha(\bar t,\bar a^\mu)\,{}_{\bar t_1}D_{\bar t}^\alpha\bar a^\nu+R_\nu^\beta(\bar t,\bar a^\mu)\times$$
$${}_{\bar t}D_{\bar t_2}^\beta\bar a^\nu-B(\bar t,\bar a^\mu))\,\mathrm{d}\bar t\qquad(36)$$

for any $[T_1,T_2]\subseteq[t_1,t_2]$

$$C_{Bf}=(R_\mu^\alpha-R_\mu^\beta)\,\xi_\mu+[(1-\alpha)R_\mu^\alpha\,{}_{t_1}D_t^\alpha a^\mu+$$
$$(1-\beta)R_\mu^\beta\,{}_tD_{t_2}^\beta a^\mu-B]\,\xi_0\qquad(37)$$

is a fractional conserved quantity for the fractional Birkhoffian system (Eq. (7)).

Therefore, for the fractional Birkhoffian system (Eq. (7)), if ξ_0^0,ξ_ν^0 satisfy Eq. (36), there exists exact invariant

$$I_{B0}=(R_\nu^\alpha-R_\nu^\beta)\,\xi_\nu^0+[(1-\alpha)R_\nu^\alpha\,{}_{t_1}D_t^\alpha a^\nu+$$

$$(1-\beta)R^{\beta}_{\nu\,t}D^{\beta}_{t_2}a^{\nu}-B]\xi^0_0 \quad (38)$$

Suppose the fractional Birkhoffian system (Eq. (7)) is disturbed by small quantities $\varepsilon Q_\mu(\mu=1,2,\cdots,2n)$, then we can get the disturbed fractional Birkhoff equations

$$\frac{\partial R^{\alpha}_{\nu}}{\partial a^{\mu}}\,{}_{t_1}D^{\alpha}_{t}a^{\nu}+\frac{\partial R^{\beta}_{\nu}}{\partial a^{\mu}}\,{}_{t}D^{\beta}_{t_2}a^{\nu}-\frac{\partial B}{\partial a^{\mu}}+{}_{t}D^{\alpha}_{t_2}R^{\alpha}_{\mu}+{}_{t_1}D^{\beta}_{t}R^{\beta}_{\mu}=-\varepsilon Q_\mu \quad (39)$$

Under the action of small forces of perturbation εQ_μ , the invariant of the system may vary. Suppose that the disturbed infinitesimal generators $\xi_\nu(\nu=1,2,\cdots,2n)$ can be expressed as

$$\xi_\nu=\xi^0_\nu+\varepsilon\xi^1_\nu+\varepsilon^2\xi^2_\nu+\cdots \quad (40)$$

Then we have Theorem 7 as follow.

Theorem 7 For the disturbed fractional Birkhoffian system (Eq. (39)), if the infinitesimal generators $\xi^j_\mu(j=0,1,2,\cdots)$ satisfy

$$\frac{\partial R^{\alpha}_{\nu}}{\partial a^{\mu}}\xi^j_\mu\cdot{}_{t_1}D^{\alpha}_{t}a^{\nu}+\frac{\partial R^{\beta}_{\nu}}{\partial a^{\mu}}\xi^j_\mu\cdot{}_{t}D^{\beta}_{t_2}a^{\nu}+R^{\alpha}_{\mu\,t_1}D^{\alpha}_{t}\xi^j_\mu+R^{\beta}_{\mu\,t}D^{\beta}_{t_2}\xi^j_\mu-\frac{\partial B}{\partial a^{\mu}}\xi^j_\mu+Q_\mu\xi^{j-1}_\mu=0 \quad (41)$$

the disturbed fractional Birkhoff system has a z-th order adiabatic invariant

$$I^B_z=\sum_{j=0}^{z}\varepsilon^j(R^{\alpha}_{\nu}-R^{\beta}_{\nu})\xi^j_\nu \quad (42)$$

where we set $\xi^{j-1}_\mu=0$, when $j=0$.

Proof From the disturbed fractional Birkhoff equations and the condition, we have

$$\sum_{j=0}^{z}\varepsilon^j[D^{\alpha}_t(R^{\alpha}_{\nu},\xi^j_\nu)-D^{\beta}_t(\xi^j_\nu,R^{\beta}_{\nu})]=\sum_{j=0}^{z}\varepsilon^j(-\xi^j_{\nu\,t}D^{\alpha}_{t_2}R^{\alpha}_{\nu}+R^{\alpha}_{\nu\,t_1}D^{\alpha}_{t}\xi^j_\nu+R^{\beta}_{\nu\,t}D^{\beta}_{t_2}\xi^j_\nu-\xi^j_{\nu\,t_1}D^{\beta}_{t}R^{\beta}_{\nu})=$$
$$\sum_{j=0}^{z}\varepsilon^j\Big(-\xi^j_{\nu\,t}D^{\alpha}_{t_2}R^{\alpha}_{\nu}-\frac{\partial R^{\alpha}_{\nu}}{\partial a^{\mu}}\xi^j_\mu\cdot{}_{t_1}D^{\alpha}_{t}a^{\nu}-\frac{\partial R^{\beta}_{\nu}}{\partial a^{\mu}}\xi^j_\mu\cdot{}_{t}D^{\beta}_{t_2}a^{\nu}+\frac{\partial B}{\partial a^{\mu}}\xi^j_\mu-Q_\mu\xi^{j-1}_\mu-\xi^j_{\nu\,t_1}D^{\beta}_{t}R^{\beta}_{\nu}\Big)=$$
$$\sum_{j=0}^{z}\varepsilon^j(-Q_\mu\xi^{j-1}_\mu+\varepsilon Q_\mu\xi^j_\mu)=\varepsilon^{z+1}Q_\mu\xi^z_\mu$$

The proof is completed.

Theorem 8 Under the infinitesimal transformations

$$\bar{t}=t+\varepsilon\xi_0(t,a^\mu)+o(\varepsilon)$$
$$\bar{a}^\nu(t)=a^\nu(t)+\varepsilon\xi_\nu(t,a^\mu)+o(\varepsilon)$$
$$\mu,\nu=1,2,\cdots,2n \quad (43)$$

where

$$\xi_0=\xi^0_0+\varepsilon\xi^1_0+\varepsilon^2\xi^2_0+\cdots$$
$$\xi_\nu=\xi^0_\nu+\varepsilon\xi^1_\nu+\varepsilon^2\xi^2_\nu+\cdots \quad (44)$$

the disturbed fractional Birkhoffian system (Eq. (39)) has a z-th order adiabatic invariant

$$I_{Bz}=\sum_{j=0}^{z}\varepsilon^j\{(R^{\alpha}_{\nu}(t,a^\mu)-R^{\beta}_{\nu}(t,a^\mu))\xi^j_\nu+[(1-\alpha)R^{\alpha}_{\nu}(t,a^\mu)\,{}_{t_1}D^{\alpha}_{t}a^{\nu}-B(t,a^\mu)+(1-\beta)R^{\beta}_{\nu}(t,a^\mu)\,{}_{t}D^{\beta}_{t_2}a^{\nu}]\xi^j_0\} \quad (45)$$

Proof Consider a one to one transformation

$$[t_1,t_2]\ni t\to f(\lambda)\in[\sigma_1,\sigma_2]$$

which satisfies $t(\sigma_1)=t_1,t(\sigma_2)=t_2$ and $t'_\sigma=\mathrm{d}t(\sigma)/\mathrm{d}\sigma=f(\lambda)=1$, when $\lambda=0$.

From the definitions of the right Riemann-Liouville fractional derivative and the left Riemann-Liouville fractional derivative, we can get

$${}_{\sigma_1}D^{\alpha}_{t(\sigma)}a^{\nu}(t(\sigma))=(t'_\sigma)^{-\alpha}\,{}_{\frac{t_1}{(t'_\sigma)^2}}D^{\alpha}_{\sigma}a^{\nu}(\sigma)$$
$${}_{t(\sigma)}D^{\beta}_{\sigma_2}a^{\nu}(t(\sigma))=(t'_\sigma)^{-\beta}\,{}_{\sigma}D^{\beta}_{\frac{t_2}{(t'_\sigma)^2}}a^{\nu}(\sigma) \quad (46)$$

Hence

$$\bar{S}(t(\cdot),a^\mu(\cdot))=\int_{\sigma_1}^{\sigma_2}[R^{\alpha}_{\nu}(t(\sigma),a^\mu(t(\sigma)))\times{}_{\sigma_1}D^{\alpha}_{t(\sigma)}a^{\nu}(t(\sigma))+R^{\beta}_{\nu}(t(\sigma),a^\mu(t(\sigma)))\,{}_{t(\sigma)}D^{\beta}_{\sigma_2}a^{\nu}(t(\sigma))-B(t(\sigma),a^\mu(t(\sigma)))]t'_\sigma\mathrm{d}\sigma=$$
$$\int_{\sigma_1}^{\sigma_2}[R^{\alpha}_{\nu}(t(\sigma),a^\mu(t(\sigma)))(t'_\sigma)^{-\alpha}\times{}_{\frac{t_1}{(t'_\sigma)^2}}D^{\alpha}_{\sigma}a^{\nu}(\sigma)+R^{\beta}_{\nu}(t(\sigma),a^\mu(t(\sigma)))\times(t'_\sigma)^{-\beta}\,{}_{\sigma}D^{\beta}_{\frac{t_2}{(t'_\sigma)^2}}a^{\nu}(\sigma)-B(t(\sigma),a^\mu(t(\sigma)))]t'_\sigma\mathrm{d}\sigma=$$
$$\int_{\sigma_1}^{\sigma_2}[R^{\alpha}_{\nu}(t(\sigma),a^\mu(t(\sigma)))(t'_\sigma)^{1-\alpha}\times{}_{\frac{t_1}{(t'_\sigma)^2}}D^{\alpha}_{\sigma}a^{\nu}(\sigma)+R^{\beta}_{\nu}(t(\sigma),a^\mu(t(\sigma)))(t'_\sigma)^{1-\beta}\,{}_{\sigma}D^{\beta}_{\frac{t_2}{(t'_\sigma)^2}}a^{\nu}(\sigma)-B(t(\sigma),a^\mu(t(\sigma)))t'_\sigma]\mathrm{d}\sigma\doteq$$
$$\int_{\sigma_1}^{\sigma_2}\bar{B}_f(t(\sigma),a^\mu(t(\sigma)),t'_\sigma,{}_{\frac{t_1}{(t'_\sigma)^2}}D^{\alpha}_{\sigma}a^{\nu}(\sigma),{}_{\sigma}D^{\beta}_{\frac{t_2}{(t'_\sigma)^2}}a^{\nu}(\sigma))\mathrm{d}\sigma=$$
$$\int_{t_1}^{t_2}\Big(R^{\alpha}_{\nu}(t,a^\mu)\,{}_{t_1}D^{\alpha}_{t}a^{\nu}+R^{\beta}_{\nu}(t,a^\mu)\,{}_{t}D^{\beta}_{t_2}a^{\nu}-B(t,a^\mu)\Big)\mathrm{d}t=S(a^\mu(\cdot))$$

For $\lambda=0$, we have

$$R_\nu^\alpha(t(\sigma),a^\mu(t(\sigma)))(t'_\sigma)^{1-\alpha}\times {}_{\frac{t_1}{(t'_\sigma)^2}}D_\sigma^\alpha a^\nu(\sigma)+R_\nu^\beta(t(\sigma),a^\mu(t(\sigma)))\times (t'_\sigma)^{1-\beta}\,{}_\sigma D^\beta_{\frac{t_2}{(t'_\sigma)^2}}a^\nu(\sigma)= R_\nu^\alpha(t,a^\mu(t))\,{}_{t_1}D_t^\alpha a^\nu(t)+ R_\nu^\beta(t,a^\mu(t))\,{}_tD_{t_2}^\beta a^\nu(t) \quad (47)$$

$$\frac{\partial \bar{B}_f}{\partial t'_\sigma}=\frac{\partial}{\partial t'_\sigma}\Big[R_\nu^\alpha(t(\sigma),a^\mu(t(\sigma)))\times (t'_\sigma)^{1-\alpha}\,{}_{\frac{t_1}{(t'_\sigma)^2}}D_\sigma^\alpha a^\nu(\sigma)+ R_\nu^\beta(t(\sigma),a^\mu(t(\sigma)))(t'_\sigma)^{1-\beta}\times {}_\sigma D^\beta_{\frac{t_2}{(t'_\sigma)^2}}a^\nu(\sigma)-B(t(\sigma),a^\mu(t(\sigma)))t'_\sigma\Big]= (1-\alpha)R^\alpha_\nu\,{}_{t_1}D_t^\alpha a^\nu+(1-\beta)R^\beta_\nu\,{}_tD^\beta_{t_2}a^\nu-B \quad (48)$$

Hence, using the similar method adopted for Theorem 4, from Theorem 7, for $\lambda=0$, we can get

$$I_{Bz}=\sum_{j=0}^{z}\varepsilon^j\left\{(R_\nu^\alpha-R_\nu^\beta)\xi_\nu^j+\frac{\partial\bar{B}_f}{\partial t'_\sigma}\cdot\xi_0^j\right\}= \sum_{j=0}^{z}\varepsilon^j\{(R_\nu^\alpha-R_\nu^\beta)\xi_\nu^j+ [(1-\alpha)R^\alpha_\nu\,{}_{t_1}D_t^\alpha a^\nu+ (1-\beta)R^\beta_\nu\,{}_tD^\beta_{t_2}a^\nu-B]\xi_0^j\}$$

The proof is completed.

3 Two Illustrative Examples

In this section, we give two examples to illustrate the results obtained above.

Example 1 Let us consider the following fractional Lagrangian system

$$L=q_1\cdot{}_aD_t^\alpha q_1\cdot{}_aD_t^\alpha q_2 \quad (49)$$

We can verify that

$$\zeta_0^1=q_1,\zeta_0^2=-2q_2 \quad (50)$$

satisfy the condition (11). Then we can obtain from Eq. (16) that

$$I_0^L=(q_1)^2\,{}_aD_t^\alpha q_2-2q_1q_2\,{}_aD_t^\alpha q_1 \quad (51)$$

Suppose the system (Eq. (5)) is disturbed by the following small quantities

$$\varepsilon Q_1=\varepsilon(-3\,{}_aD_t^\alpha q_1\cdot{}_aD_t^\alpha q_2),\ \varepsilon Q_2=0 \quad (52)$$

By calculating, the following solutions

$$\zeta_1^1=q_1,\zeta_1^2=q_2 \quad (53)$$

satisfy Eq. (20). Therefore, from Theorem 3, we get

$$I_1^L=(q_1)^2\,{}_aD_t^\alpha q_2-2q_1q_2\,{}_aD_t^\alpha q_1+ \varepsilon[(q_1)^2\,{}_aD_t^\alpha q_2+q_1q_2\,{}_aD_t^\alpha q_1] \quad (54)$$

Of course, we can also obtain the higher-order adiabatic invariants.

Example 2 Let us consider the extreme value for the following fractional problem of the calculus of variations

$$S(a^\mu(\cdot))=\int_{t_1}^{t_2}(a^2\,{}_{t_1}D_t^\alpha a^1+a^4\,{}_{t_1}D_t^\alpha a^3-a^2a^3)\,\mathrm{d}t \quad (55)$$

The problem (Eq. (55)) is a fourth order Pfaff-Birkhoff fractional problem of the calculus of variations in terms of Riemann-Liouville derivatives. From Eq. (55), we obtain that

$$B=a^2a^3,R_1^\alpha=a^2,R_3^\alpha=a^4,$$
$$R_2^\alpha=R_4^\alpha=0,R_j^\beta=0\quad j=1,2,3,4 \quad (56)$$

Obviously, the following solutions

$$\xi_1^0=a^1,\xi_2^0=-a^2$$
$$\xi_3^0=a^3,\xi_4^0=-a^4 \quad (57)$$

satisfy the condition (32). Then we can get the exact invariant from Eq. (34) that

$$I_0^B=a^1a^2+a^3a^4 \quad (58)$$

Suppose the system (Eq. (7)) is disturbed by the following small quantities

$$\varepsilon Q_1=\varepsilon a^2,\varepsilon Q_2=\varepsilon Q_3=0,\varepsilon Q_4=\varepsilon a^3 \quad (59)$$

By some calculations, the following solutions

$$\xi_1^1=0,\xi_2^1=-a^1,\xi_3^1=a^3$$
$$\xi_4^1=-a^4,\xi_5^1=a^2,\xi_6^1=0 \quad (60)$$

satisfy Eq. (41). Hence, from Theorem 7, we get

$$I_1^B=a^1a^2+a^3a^4+\varepsilon(a^2a^3+a^1a^4) \quad (61)$$

Of course, we can also obtain the higher-order adiabatic invariants.

4 Conclusions

In this paper, adiabatic invariants are studied for the fractional Lagrangian system and the fractional Birkhoffian system in the sense of Riemann-Liouville derivatives under the special and general infinitesimal transformations. We can also get adiabatic invariants in the sense of Caputo derivatives, Riesz-Caputo derivatives, Riesz-Riemann-Liouville derivatives and so on. Besides, much work deserves to do since adiabatic invariant and fractional variational problems are still in their early days.

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