Noether Theorem for Generalized Birkhoffian Systems with Time Delay

Zhai Xianghua[1], *Zhang Yi*[2*]

1. School of Science, Nanjing University of Science and Technology, Nanjing 210094, P. R. China;
2. School of Civil Engineering, Suzhou University of Science and Technology, Suzhou 215009, P. R. China

(Received 28 July 2016, revised 13 October 2016, accepted 20 December 2016)

Abstract: The Noether symmetries and the conserved quantities for generalized Birkhoffian systems with time delay are studied. Firstly, the generalized Pfaff-Birkhoff principle with time delay is proposed, and the generalized Birkhoff's equations with time delay are obtained. Secondly, the generalized Noether quasi-symmetric transformations of the system are defined, and the criterion of the Noether symmetries is established. Then the Noether theorem for generalized Birkhoffian systems with time delay is established. Finally, by imposing restrictions of constraints on the infinitesimal transformations, the Noether theorem of constrained Birkhoffian systems with time delay is established. One example is given to illustrate the application of the results.

Key words: time delay; generalized Birkhoffian system; Noether symmetry; conserved quantity

CLC number: O316 **Document code**: A **Article ID**: 1005-1120(2018)03-0507-09

0 Introduction

In recent years, time-delay phenomena in real-life are widely noted and studied. Especially in the process of application of computer control technology, sensor testing technology and vibration control technology in various engineering fields, even exiting a time-delay with millisecond leads to complex changes of the stability and control performance of the system[1-4]. And the research subject on symmetry and conserved quantity has always been paid close attention by researchers in mathematics, mechanics and physics. It is still been an expansive area studying on the symmetries and conserved quantities when considering the influence of time delay for mechanical systems.

As an important theoretical basis and mathematical method, the variational problems with time delay have been studied. Él′sgol′c[5] was the first researcher who proposed the variational problems with delayed arguments. And after that, many results to this kind of research topic combining with multifarious practical situations are made[6-11]. It is note worthy that the Noether symmetries for variational and optimal problems with time delay were studied by Frederico and Torres[12] firstly. Recently, Zhang and Jin[13-14] studied the Noether theory for non-conservative dynamical systems with time delay and the Hamiltonian mechanical systems with time delay, and derived the relevant results to the fractional model[15-16].

In 1927, the American mathematician Birkhoff[17] presented a kind of more general dynamic equations (Birkhoff's equations) and a more general integral variational principle (Pfaff-Birkhoff principle). Until now, many research branches like the approaches of integration and reduction, the stability of motion and the dynamical inverse problem of Birkhoffian mechanics as well as the practical applications have made a great progress[18-19]. Mei[20-21] established the Noether theory for generalized Birkhoffian systems and con-

* Corresponding author, E-mail address: zhy@mail.usts.edu.cn.

How to cite this article: Zhai Xianghua, Zhang Yi. Noether theorem for generalized Birkhoffian systems with time delay [J]. Trans. Nanjing Univ. Aero. Astro., 2018,35(3):507-515.
http://dx.doi.org/10.16356/j.1005-1120.2018.03.507

strained Birkhoffian systems. In the recent paper[22], the Noether symmetries and conserved quantities for Birkhoffian systems with time delay were studied. However, the results in Ref. [22] can not be generalized to the generalized Birkhoffian systems with time delay directly. Moreover, the constrained Birkhoffian systems in the sense of time delay have not been investigated yet. In view of the development trends above, it is necessary and meaningfully to study these new problems.

1 Generalized Birkhoff's Equations with Time Delay

We review some known results in the literature about standard generalized Birkhoff's equations[19] without considering the influence of time delay.

The generalized Pfaff-Birkhoff principle can be expressed as

$$\int_{t_1}^{t_2}[\delta(R_\nu(t,\boldsymbol{a})\dot{a}^\nu - B(t,\boldsymbol{a})) + \delta' W]\,\mathrm{d}t = 0 \quad (1)$$

with the commutative conditions

$$\mathrm{d}\delta a^\nu = \delta \mathrm{d}a^\nu \quad \nu = 1,2,\cdots,2n \quad (2)$$

and the boundary conditions

$$\delta a^\nu\big|_{t=t_1} = \delta a^\nu\big|_{t=t_2} = 0 \quad \nu = 1,2,\cdots,2n \quad (3)$$

where $B(t,\boldsymbol{a})$ is the Birkhoffian, $R_\nu(t,\boldsymbol{a})$ are Birkhoff's functions, $\delta' W = \Lambda_\nu(t,\boldsymbol{a})\delta a^\nu$, and the arbitrary differentiable functions $\Lambda_\nu(t,\boldsymbol{a})$ are called additional items. When $\delta' W = 0$, the principle (1) is reduced to the standard Pfaff-Birkhoff-principle[18].

From the principle (1) we can derive the standard generalized Birkhoff's equations

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right) = -\Lambda_\mu$$
$$\mu = 1,2,\cdots,2n \quad (4)$$

Now, we consider a Birkhoffian system with time delay whose Birkhoffian and Birkhoff's functions are as follows

$$B(t,\boldsymbol{a}(t),\boldsymbol{a}(t-\tau)) \triangleq B(t,\boldsymbol{a},\boldsymbol{a}_\tau)$$
$$R_\mu(t,\boldsymbol{a}(t)) \triangleq R_\mu(t,\boldsymbol{a})$$
$$R_\mu(t,\boldsymbol{a}(t-\tau)) \triangleq R_{\mu\tau}(t,\boldsymbol{a}_\tau) \quad (5)$$

First, the generalized Pfaff-Birkhoff principle with time delay can be established as

$$\int_{t_1}^{t_2}[\delta(R_\nu(t,\boldsymbol{a})\dot{a}^\nu + R_{\nu\tau}(t,\boldsymbol{a}_\tau)\dot{a}_\tau^\nu - B(t,\boldsymbol{a},\boldsymbol{a}_\tau)) + \delta' W']\,\mathrm{d}t = 0 \quad (6)$$

where $\delta' W' = \Lambda'_\nu(t,\boldsymbol{a},\boldsymbol{a}_\tau)\delta a^\nu, \nu = 1,2,\cdots,2n$.

Moreover, the principle (6) satisfies the commutative condition

$$\mathrm{d}\delta a^\nu = \delta \mathrm{d}a^\nu \quad \nu = 1,2,\cdots,2n \quad (7)$$

and the boundary conditions

$$a^\nu(t) = f_\nu(t)$$
$$t \in [t_1 - \tau, t_1] \quad \nu = 1,2,\cdots,2n \quad (8)$$

$$a^\nu(t) = a^\nu(t_2) \quad t = t_2,\ \nu = 1,2,\cdots,2n \quad (9)$$

where τ is a given positive real number such that $\tau < t_2 - t_1$, and $f_\nu(t)$ are given piecewise smooth functions in the interval $[t_1 - \tau, t_1]$, then the principle (6) can be written as

$$\int_{t_1}^{t_2}\left[\frac{\partial R_\nu}{\partial a^\mu}\delta a^\mu \dot{a}^\nu + R_\nu \delta\dot{a}^\nu + \frac{\partial R_{\nu\tau}}{\partial a_\tau^\mu}\delta a_\tau^\mu \dot{a}_\tau^\nu + R_{\nu\tau}\delta\dot{a}_\tau^\nu - \left(\frac{\partial B}{\partial a^\mu}\delta a^\mu + \frac{\partial B}{\partial a_\tau^\mu}\delta a_\tau^\mu\right) + \Lambda'_\mu \delta a^\mu\right]\mathrm{d}t = 0 \quad (10)$$

By integrating by parts and performing a linear change of variables $t = \theta + \tau$ and noticing the boundary conditions (8) and (9), Eq. (10) can be written as

$$-\left[\delta a^\mu \int_t^{t_2-\tau}\left(\frac{\partial R_\nu}{\partial a^\mu}(\theta)\dot{a}^\nu(\theta) - \frac{\partial B}{\partial a^\mu}(\theta) + \frac{\partial R_{\nu\tau}}{\partial a_\tau^\mu}(\theta+\tau)\dot{a}_\tau^\nu(\theta+\tau) - \frac{\partial B}{\partial a_\tau^\mu}(\theta+\tau) + \Lambda'_\mu(\theta)\right)\mathrm{d}\theta\right]\Bigg|_{t_1}^{t_2-\tau} +$$
$$\int_{t_1}^{t_2-\tau}\delta\dot{a}^\mu\left[\int_t^{t_2-\tau}\left(\frac{\partial R_\nu}{\partial a^\mu}(\theta)\dot{a}^\nu(\theta) - \frac{\partial B}{\partial a^\mu}(\theta) + \frac{\partial R_{\nu\tau}}{\partial a_\tau^\mu}(\theta+\tau)\dot{a}_\tau^\nu(\theta+\tau) - \frac{\partial B}{\partial a_\tau^\mu}(\theta+\tau) + \Lambda'_\mu(\theta)\right)\mathrm{d}\theta + R_\mu(t) + R_{\mu\tau}(t+\tau)\right]\mathrm{d}t +$$
$$\left[\delta a^\mu \int_{t_2-\tau}^t\left(\frac{\partial R_\nu}{\partial a^\mu}(\theta)\dot{a}^\nu(\theta) - \frac{\partial B}{\partial a^\mu}(\theta) + \Lambda'_\mu(\theta)\right)\mathrm{d}\theta\right]\Bigg|_{t_2-\tau}^{t_2} -$$
$$\int_{t_2-\tau}^{t_2}\delta\dot{a}^\mu\left[\int_{t_2-\tau}^t\left(\frac{\partial R_\nu}{\partial a^\mu}(\theta)\dot{a}^\nu(\theta) - \frac{\partial B}{\partial a^\mu}(\theta) + \Lambda'_\mu(\theta)\right)\mathrm{d}\theta - R_\mu(t)\right]\mathrm{d}t =$$
$$\int_{t_1}^{t_2-\tau}\delta\dot{a}^\mu\left[\int_t^{t_2-\tau}\left(\frac{\partial R_\nu}{\partial a^\mu}(\theta)\dot{a}^\nu(\theta) - \frac{\partial B}{\partial a^\mu}(\theta) + \frac{\partial R_{\nu\tau}}{\partial a_\tau^\mu}(\theta+\tau)\dot{a}_\tau^\nu(\theta+\tau) - \frac{\partial B}{\partial a_\tau^\mu}(\theta+\tau) + \right.\right.$$

$$\Lambda'_{\mu}(\theta))\,\mathrm{d}\theta + R_{\mu}(t) + R_{\mu\tau}(t+\tau)\Big]\mathrm{d}t - \int_{t_2-\tau}^{t_2}\delta\dot{a}^{\mu}\Big[\int_{t_2-\tau}^{t}\Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}(\theta)\dot{a}^{\nu}(\theta) - \frac{\partial B}{\partial a^{\mu}}(\theta) + \Lambda'_{\mu}(\theta)\Big)\mathrm{d}\theta - R_{\mu}(t)\Big]\mathrm{d}t = 0 \quad (11)$$

Since the variation $\delta\dot{a}^{\nu}$ are independent of each other, therefore, by the fundamental lemma[23] of the calculus of variations, we can derive

$$\Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}(t) - \frac{\partial R_{\mu}}{\partial a^{\nu}}(t)\Big)\dot{a}^{\nu}(t) - \Big(\frac{\partial B}{\partial a^{\mu}}(t) + \frac{\partial R_{\mu}}{\partial t}(t)\Big) + \Big(\frac{\partial R_{\nu\tau}}{\partial a_{\tau}^{\mu}}(t+\tau) - \frac{\partial R_{\mu\tau}}{\partial a_{\tau}^{\nu}}(t+\tau)\Big)\dot{a}_{\tau}^{\nu}(t+\tau) - \Big(\frac{\partial B}{\partial a_{\tau}^{\mu}}(t+\tau) + \frac{\partial R_{\mu\tau}}{\partial t}(t+\tau)\Big) = -\Lambda'_{\mu}(t)$$
$$t \in [t_1, t_2-\tau]$$
$$\Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}(t) - \frac{\partial R_{\mu}}{\partial a^{\nu}}(t)\Big)\dot{a}^{\nu}(t) - \Big(\frac{\partial B}{\partial a^{\mu}}(t) + \frac{\partial R_{\mu}}{\partial t}(t)\Big) = -\Lambda'_{\mu}(t) \quad t \in (t_2-\tau, t_2] \quad (12)$$

Eq. (12) can be called the differential equations of motion of the generalized Birkhoffian system with time delay. If time delay does not exist, Eq. (12) is reduced to standard generalized Birkhoff's Eq. (4).

2 Variation of Pfaff Action with Time Delay

Introduce the infinitesimal transformations of r-parameter finite transformation group G_r

$$t^* = t + \Delta t, a^{\mu *} = a^{\mu} + \Delta a^{\mu} \quad \mu = 1, 2, \cdots, 2n \quad (13)$$

and their expanding formulae are

$$t^* = t + \varepsilon_{\alpha}\xi_0^{\alpha}(t, \boldsymbol{a}), a^{\mu *} = a^{\mu} + \varepsilon_{\alpha}\xi_{\mu}^{\alpha}(t, \boldsymbol{a}) \quad \mu = 1, 2, \cdots, 2n \quad (14)$$

where ε_{α} ($\alpha = 1, 2, \cdots, r$) are infinitesimal parameters, ξ_0^{α} and ξ_{μ}^{α} are called the infinitesimal generators or the generating functions of the infinitesimal transformations.

The Pfaff action with time delay in Ref. [22] is expressed as

$$A' = \int_{t_1}^{t_2}(R_{\nu}(t, \boldsymbol{a})\mathrm{d}a^{\nu} + R_{\nu\tau}(t, \boldsymbol{a}_{\tau})\mathrm{d}a_{\tau}^{\nu} - B(t, \boldsymbol{a}, \boldsymbol{a}_{\tau})\mathrm{d}t) \quad (15)$$

The variation of Pfaff action with time delay was discussed in Ref. [22] and two basic formulae were obtained as follows

$$\Delta A' = \int_{t_1}^{t_2-\tau}\varepsilon_{\alpha}\Big\{\frac{\mathrm{d}}{\mathrm{d}t}\Big[R_{\nu}(t)\xi_{\nu}^{\alpha} + R_{\nu\tau}(t+\tau)\xi_{\nu}^{\alpha} - B(t)\xi_0^{\alpha}\Big] + \Big[\Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}(t) - \frac{\partial R_{\mu}}{\partial a^{\nu}}(t)\Big)\dot{a}^{\nu}(t) - \Big(\frac{\partial B}{\partial a^{\mu}}(t) + \frac{\partial R_{\mu}}{\partial t}(t)\Big) + \Big(\frac{\partial R_{\nu\tau}}{\partial a_{\tau}^{\mu}}(t+\tau) - \frac{\partial R_{\mu\tau}}{\partial a_{\tau}^{\nu}}(t+\tau)\Big)\dot{a}_{\tau}^{\nu}(t+\tau) - \Big(\frac{\partial B}{\partial a_{\tau}^{\mu}}(t+\tau) + \frac{\partial R_{\mu\tau}}{\partial t}(t+\tau)\Big)\Big]\Big(\xi_{\mu}^{\alpha} - \dot{a}^{\mu}(t)\xi_0^{\alpha}\Big)\Big\}\mathrm{d}t + \int_{t_2-\tau}^{t_2}\varepsilon_{\alpha}\Big\{\frac{\mathrm{d}}{\mathrm{d}t}[R_{\nu}(t)\xi_{\nu}^{\alpha} - B(t)\xi_0^{\alpha}] + \Big[\Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}(t) - \frac{\partial R_{\mu}}{\partial a^{\nu}}(t)\Big)\dot{a}^{\nu}(t) - \Big(\frac{\partial B}{\partial a^{\mu}}(t) + \frac{\partial R_{\mu}}{\partial t}(t)\Big)\Big](\xi_{\mu}^{\alpha} - \dot{a}^{\mu}(t)\xi_0^{\alpha})\Big\}\mathrm{d}t \quad (16)$$

and

$$\Delta A' = \int_{t_1}^{t_2-\tau}\Big[\Big(R_{\nu}(t)\dot{a}^{\nu}(t) + R_{\nu\tau}(t+\tau)\dot{a}_{\tau}^{\nu}(t+\tau) - B(t)\Big)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + \Big(\frac{\partial R_{\nu}}{\partial t}(t)\dot{a}^{\nu}(t) + \frac{\partial R_{\nu\tau}}{\partial t}(t+\tau)\dot{a}_{\tau}^{\nu}(t+\tau) - \frac{\partial B}{\partial t}(t)\Big)\Delta t + \Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}(t)\dot{a}^{\nu}(t) - \frac{\partial B}{\partial a^{\mu}}(t)\Big)\Delta a^{\mu} + \Big(\frac{\partial R_{\nu\tau}}{\partial a_{\tau}^{\mu}}(t+\tau)\dot{a}_{\tau}^{\nu}(t+\tau) - \frac{\partial B}{\partial a_{\tau}^{\mu}}(t+\tau)\Big)\Delta a^{\mu} + R_{\nu}(t)\Delta\dot{a}^{\nu} + R_{\nu\tau}(t+\tau)\Delta\dot{a}^{\nu}\Big]\mathrm{d}t + \int_{t_2-\tau}^{t_2}\Big[(R_{\nu}(t)\dot{a}^{\nu}(t) - B(t))\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + \Big(\frac{\partial R_{\nu}}{\partial t}(t)\dot{a}^{\nu}(t) - \frac{\partial B}{\partial t}(t)\Big)\Delta t + \Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}(t)\dot{a}^{\nu}(t) - \frac{\partial B}{\partial a^{\mu}}(t)\Big)\Delta a^{\mu} + R_{\nu}(t)\Delta\dot{a}^{\nu}\Big]\mathrm{d}t \quad (17)$$

3 Noether Symmetries with Time Delay

Now, we give the definitions of the Noether symmetric transformations in time-delay situation.

Definition 1[22] If the Pfaff action (15) is invariant under the infinitesimal transformations (13) of group, i. e., for each of the infinitesimal transformations, the formula

$$\Delta A' = 0 \quad (18)$$

holds, then the infinitesimal transformations are called Noether symmetric transformations.

Definition 2[22] If the Pfaff action (15) is quasi-invariant under the infinitesimal transformations (13) of group, i. e. , for each of the infinitesimal transformations, the formula

$$\Delta A' = -\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\,\mathrm{d}t \tag{19}$$

holds, where $\Delta G = \varepsilon_\alpha G^\alpha$, and $G^\alpha = G^\alpha(t, \boldsymbol{a}, \boldsymbol{a}_\tau)$ is the gauge function, then the infinitesimal transformations are called Noether quasi-symmetric transformations.

Definition 3 If the Pfaff action (15) is generalized quasi-invariant under the infinitesimal transformations (13) of group, i. e. , for each of the infinitesimal transformations, the formula

$$\Delta A' = -\int_{t_1}^{t_2} \left[\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G) + \Lambda'_\mu \delta a^\mu\right]\mathrm{d}t \tag{20}$$

holds, where $\Delta G = \varepsilon_\alpha G^\alpha$, and $G^\alpha = G^\alpha(t, \boldsymbol{a}, \boldsymbol{a}_\tau)$ is the gauge function, and $\Lambda'_\mu = \Lambda'_\mu(t, \boldsymbol{a}, \boldsymbol{a}_\tau)$, then the infinitesimal transformations are called generalized Noether quasi-symmetric transformations.

According to Definition 3 and Eq. (17), we can yield the following criterion.

Criterion 1 If the infinitesimal transformations (14) of group satisfy the following conditions

$$\begin{gathered}
(R_\nu(t)\dot{a}^\nu(t) + R_{\nu\tau}(t+\tau)\dot{a}^\nu(t) - B(t))\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + \\
\Big(\frac{\partial R_\nu}{\partial t}(t)\dot{a}^\nu(t) + \frac{\partial R_{\nu\tau}}{\partial t}(t+\tau)\dot{a}^\nu(t) - \\
\frac{\partial B}{\partial t}(t) - \Lambda'_\nu(t)\dot{a}^\nu(t)\Big)\Delta t + \\
\Big(\frac{\partial R_\nu}{\partial a^\mu}(t)\dot{a}^\nu(t) - \frac{\partial B}{\partial a^\mu}(t) + \Lambda'_\mu(t)\Big)\Delta a^\mu + \\
\Big(\frac{\partial R_{\nu\tau}}{\partial a^\mu_\tau}(t+\tau)\dot{a}^\nu_\tau(t+\tau) - \frac{\partial B}{\partial a^\mu_\tau}(t+\tau)\Big)\Delta a^\mu + \\
R_\nu(t)\Delta\dot{a}^\nu + R_{\nu\tau}(t+\tau)\Delta\dot{a}^\nu = -\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G) \\
t \in [t_1, t_2 - \tau] \\
(R_\nu(t)\dot{a}^\nu(t) - B(t))\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + \\
\Big(\frac{\partial R_\nu}{\partial t}(t)\dot{a}^\nu(t) - \frac{\partial B}{\partial t}(t) - \Lambda'_\nu(t)\dot{a}^\nu(t)\Big)\Delta t + \\
\Big(\frac{\partial R_\nu}{\partial a^\mu}(t)\dot{a}^\nu(t) - \frac{\partial B}{\partial a^\mu}(t) + \Lambda'_\mu(t)\Big)\Delta a^\mu + \\
R_\nu(t)\Delta\dot{a}^\nu = -\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G) \quad t \in (t_2 - \tau, t_2]
\end{gathered} \tag{21}$$

then the transformations (13) are the generalized Noether quasi-symmetric transformations for the generalized Birkhoffian system with time delay.

Furthermore, in consideration of the expanding formulae (14) of the infinitesimal transformations (13), formula (21) can be expressed as

$$\begin{gathered}
R_\nu(t)\dot{\xi}^\alpha_\nu + R_{\nu\tau}(t+\tau)\dot{\xi}^\alpha_\nu - B(t)\dot{\xi}^\alpha_0 + \\
\Big(\frac{\partial R_\nu}{\partial t}(t)\dot{a}^\nu(t) + \frac{\partial R_{\nu\tau}}{\partial t}(t+\tau)\dot{a}^\nu_\tau(t+\tau) - \\
\frac{\partial B}{\partial t}(t) - \Lambda'_\nu(t)\dot{a}^\nu(t)\Big)\xi^\alpha_0 + \\
\Big(\frac{\partial R_\nu}{\partial a^\mu}(t)\dot{a}^\nu(t) - \frac{\partial B}{\partial a^\mu}(t) + \Lambda'_\mu(t)\Big)\xi^\alpha_\mu + \\
\Big(\frac{\partial R_{\nu\tau}}{\partial a^\mu_\tau}(t+\tau)\dot{a}^\nu_\tau(t+\tau) - \frac{\partial B}{\partial a^\mu_\tau}(t+\tau)\Big)\xi^\alpha_\mu = -\dot{G}^\alpha \\
t \in [t_1, t_2 - \tau] \\
R_\nu(t)\dot{\xi}^\alpha_\nu - B(t)\dot{\xi}^\alpha_0 + \\
\Big(\frac{\partial R_\nu}{\partial t}(t)\dot{a}^\nu(t) - \frac{\partial B}{\partial t}(t) - \Lambda'_\nu(t)\dot{a}^\nu(t)\Big)\xi^\alpha_0 + \\
\Big(\frac{\partial R_\nu}{\partial a^\mu}(t)\dot{a}^\nu(t) - \frac{\partial B}{\partial a^\mu}(t) + \Lambda'_\mu(t)\Big)\xi^\alpha_\mu = -\dot{G}^\alpha \\
t \in (t_2 - \tau, t_2]
\end{gathered} \tag{22}$$

where $\alpha = 1, 2, \cdots, r$.

When $r = 1$, Eq. (22) can be called the Noether identities of the generalized Birkhoffian system with time delay. Especially, if the time delay does not exist, Criterion 1 is reduced to the criterion of the Noether symmetries for standard generalized Birkhoffian system.

4 Noether Theorem with Time Delay

Now we give the following Noether theorem in which the conserved quantities are derived from the generalized Noether quasi-symmetries of the generalized Birkhoffian system (12) with time delay.

Theorem 1 For the generalized Birkhoffian system (12) with time delay, if the infinitesimal transformations (14) satisfy the conditions (22), then the system (12) has the conserved quantities of the following form

$$\begin{gathered}
I^\alpha = R_\nu(t)\xi^\alpha_\nu + R_{\nu\tau}(t+\tau)\xi^\alpha_\nu - \\
B(t)\xi^\alpha_0 + G^\alpha = c^\alpha \quad t \in [t_1, t_2 - \tau] \\
I^\alpha = R_\nu(t)\xi^\alpha_\nu - B(t)\xi^\alpha_0 + G^\alpha = c^\alpha \\
t \in (t_2 - \tau, t_2]
\end{gathered} \tag{23}$$

where $\alpha = 1, 2, \cdots, r$.

Proof Note that, the infinitesimal transformations (14) are the generalized Noether quasi-symmetric transformations of the system (12). According to Criterion 1, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[R_\nu(t)\xi_\nu^\alpha + R_{\nu\tau}(t+\tau)\xi_\nu^\alpha - B(t)\xi_0^\alpha + G^\alpha] + \Big[\Big(\frac{\partial R_\nu}{\partial a^\mu}(t) - \frac{\partial R_\mu}{\partial a^\nu}(t)\Big)\dot a^\nu(t) - \Big(\frac{\partial B}{\partial a^\mu}(t) + \frac{\partial R_\mu}{\partial t}(t)\Big) + \Big(\frac{\partial R_{\nu\tau}}{\partial a_\tau^\mu}(t+\tau) - \frac{\partial R_{\mu\tau}}{\partial a_\tau^\nu}(t+\tau)\Big)\dot a_\tau^\nu(t+\tau) - \Big(\frac{\partial B}{\partial a_\tau^\mu}(t+\tau) + \frac{\partial R_{\mu\tau}}{\partial t}(t+\tau)\Big) + \Lambda'_\mu(t)\Big](\xi_\mu^\alpha - \dot a^\mu(t)\xi_0^\alpha) = 0 \quad t\in[t_1, t_2-\tau]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[R_\nu(t)\xi_\nu^\alpha - B(t)\xi_0^\alpha + G^\alpha] + \Big[\Big(\frac{\partial R_\nu}{\partial a^\mu}(t) - \frac{\partial R_\mu}{\partial a^\nu}(t)\Big)\dot a^\nu(t) - \Big(\frac{\partial B}{\partial a^\mu}(t) + \frac{\partial R_\mu}{\partial t}(t)\Big) + \Lambda'_\mu(t)\Big](\xi_\mu^\alpha - \dot a^\mu(t)\xi_0^\alpha) = 0 \quad t\in(t_2-\tau, t_2]$$

where $\alpha = 1,2,\cdots,r$. Noticing Eq. (12), we can prove the theorem easily.

Theorem 1 is called the Noether theorem of the generalized Birkhoffian system with time delay. Especially, if the time delay does not exist, the Noether theorem of the generalized Birkhoffian system with time delay is reduced to the Noether theorem of standard generalized Birkhoffian system[19].

5 Noether Theorem of Constrained Birkhoffian Systems with Time Delay

Next, we study the Noether theorem of constrained Birkhoffian systems with time delay.

Assume that the motion of the Birkhoffian system with time delay is subjected to the following g bilateral ideal constraints

$$f_\beta(t, a^\mu) = 0 \quad \beta = 1,2,\cdots,g \tag{24}$$

by taking the isochronal variation of Eq. (24), we have

$$\frac{\partial f_\beta}{\partial a^\mu}\delta a^\mu = 0 \quad \beta = 1,2,\cdots,g \tag{25}$$

The Pfaff-Birkhoff principle with time delay[22] can be expressed as

$$\delta\int_{t_1}^{t_2}[R_\nu(t,\boldsymbol{a})\dot a^\nu + R_{\nu\tau}(t,\boldsymbol{a}_\tau)\dot a_\tau^\nu - B(t,\boldsymbol{a},\boldsymbol{a}_\tau)]\mathrm{d}t = 0 \tag{26}$$

Introducing the Lagrange's multipliers λ_β, we can derive the equations of motion of the constrained Birkhoffian system with time delay by combining Eqs. (25), (26), which are

$$\Big(\frac{\partial R_\nu}{\partial a^\mu}(t) - \frac{\partial R_\mu}{\partial a^\nu}(t)\Big)\dot a^\nu(t) - \Big(\frac{\partial B}{\partial a^\mu}(t) + \frac{\partial R_\mu}{\partial t}(t)\Big) + \Big(\frac{\partial R_{\nu\tau}}{\partial a_\tau^\mu}(t+\tau) - \frac{\partial R_{\mu\tau}}{\partial a_\tau^\nu}(t+\tau)\Big)\dot a_\tau^\nu(t+\tau) - \Big(\frac{\partial B}{\partial a_\tau^\mu}(t+\tau) + \frac{\partial R_{\mu\tau}}{\partial t}(t+\tau)\Big) = \lambda_\beta(t)\frac{\partial f_\beta}{\partial a^\mu}(t) \quad t\in[t_1, t_2-\tau]$$

$$\Big(\frac{\partial R_\nu}{\partial a^\mu}(t) - \frac{\partial R_\mu}{\partial a^\nu}(t)\Big)\dot a^\nu(t) - \Big(\frac{\partial B}{\partial a^\mu}(t) + \frac{\partial R_\mu}{\partial t}(t)\Big) = \lambda_\beta(t)\frac{\partial f_\beta}{\partial a^\mu}(t) \quad t\in(t_2-\tau, t_2] \tag{27}$$

Combining Eq. (24) with Eq. (27), we can find λ_β as the functions of t, a^μ, a_τ^μ. Therefore, Eq. (27) can be written as

$$\Big(\frac{\partial R_\nu}{\partial a^\mu}(t) - \frac{\partial R_\mu}{\partial a^\nu}(t)\Big)\dot a^\nu(t) - \Big(\frac{\partial B}{\partial a^\mu}(t) + \frac{\partial R_\mu}{\partial t}(t)\Big) + \Big(\frac{\partial R_{\nu\tau}}{\partial a_\tau^\mu}(t+\tau) - \frac{\partial R_{\mu\tau}}{\partial a_\tau^\nu}(t+\tau)\Big)\dot a_\tau^\nu(t+\tau) - \Big(\frac{\partial B}{\partial a_\tau^\mu}(t+\tau) + \frac{\partial R_{\mu\tau}}{\partial t}(t+\tau)\Big) = P_\mu(t) \quad t\in[t_1, t_2-\tau]$$

$$\Big(\frac{\partial R_\nu}{\partial a^\mu}(t) - \frac{\partial R_\mu}{\partial a^\nu}(t)\Big)\dot a^\nu(t) - \Big(\frac{\partial B}{\partial a^\mu}(t) + \frac{\partial R_\mu}{\partial t}(t)\Big) = P_\mu(t) \quad t\in(t_2-\tau, t_2] \tag{28}$$

where $P_\mu = P_\mu(t,\boldsymbol{a},\boldsymbol{a}_\tau) = \lambda_\beta\frac{\partial f_\beta}{\partial a^\mu}$.

Eq. (28) are called the equations of motion of the free Birkhoffian system with time delay which corresponds to the constrained Birkhoffian system with time delay, that is, the equations of motion of the corresponding free Birkhoffian system with time delay. If the initial conditions of the motion satisfy the constrained conditions (24), then the solution of the corresponding free system (28) will give the motion of the constrained Birkhoffian system with time delay.

We observe that Eq. (28) of the corresponding free Birkhoffian system with time delay are in accordance with the generalized Birkhoffian system (12) with time delay. Just take $P_\mu = -\Lambda'_\mu$.

Therefore, Theorem 1 can be applied in the corresponding free Birkhoffian system (28) with time delay.

Theorem 2 For the corresponding free Birkhoffian system (28) with time delay, if the infinitesimal transformations (14) satisfy the conditions

$$R_\nu(t)\dot{\xi}_\nu^\alpha + R_{\nu\tau}(t+\tau)\dot{\xi}_\nu^\alpha - B(t)\dot{\xi}_0^\alpha + \Big(\frac{\partial R_\nu}{\partial t}(t)\dot{a}^\nu(t) + \frac{\partial R_{\nu\tau}}{\partial t}(t+\tau)\dot{a}_\tau^\nu(t+\tau) - \frac{\partial B}{\partial t}(t) + P_\mu(t)\dot{a}^\nu(t)\Big)\xi_0^\alpha + \Big(\frac{\partial R_\nu}{\partial a^\mu}(t)\dot{a}^\nu(t) - \frac{\partial B}{\partial a^\mu}(t) - P_\mu(t)\Big)\xi_\mu^\alpha + \Big(\frac{\partial R_{\nu\tau}}{\partial a_\tau^\mu}(t+\tau)\dot{a}_\tau^\nu(t+\tau) - \frac{\partial B}{\partial a_\tau^\mu}(t+\tau)\Big)\xi_\mu^\alpha = -\dot{G}^\alpha$$
$$t \in [t_1, t_2-\tau]$$
$$R_\nu(t)\dot{\xi}_\nu^\alpha - B(t)\dot{\xi}_0^\alpha + \Big(\frac{\partial R_\nu}{\partial t}(t)\dot{a}^\nu(t) - \frac{\partial B}{\partial t}(t) + P_\mu(t)\dot{a}^\nu(t)\Big)\xi_0^\alpha + \Big(\frac{\partial R_\nu}{\partial a^\mu}(t)\dot{a}^\nu(t) - \frac{\partial B}{\partial a^\mu}(t) - P_\mu(t)\Big)\xi_\mu^\alpha = -\dot{G}^\alpha$$
$$t \in (t_2-\tau, t_2] \tag{29}$$

where $\alpha = 1,2,\cdots,r$, then the system (28) has the conserved quantities of the form (23).

Theorem 2 can be called the generalized Noether theorem of the corresponding free Birkhoffian system with time delay.

Eq. (25) can be expressed as

$$\frac{\partial f_\beta}{\partial a^\mu}(\Delta a^\mu - \dot{a}^\mu \Delta t) = \varepsilon_\alpha \frac{\partial f_\beta}{\partial a^\mu}(\xi_\mu^\alpha - \dot{a}^\mu \xi_0^\alpha) = 0$$
$$\beta = 1,2,\cdots,g;\alpha = 1,2,\cdots,r \tag{30}$$

Considering the independence of ε_α, we have

$$\frac{\partial f_\beta}{\partial a^\mu}(\xi_\mu^\alpha - \dot{a}^\mu \xi_0^\alpha) = 0$$
$$\beta = 1,2,\cdots,g;\alpha = 1,2,\cdots,r \tag{31}$$

Eq. (31) is the restrictions of constraints on the infinitesimaltransformations.

Then, we can establish the Noether theorem for the constrained Birkhoffian system with time delay.

Theorem 3 For given constrained Birkhoffian systems (24) and (27) with time delay, if the infinitesimal transformations (14) satisfy the conditions

$$R_\nu(t)\dot{\xi}_\nu^\alpha + R_{\nu\tau}(t+\tau)\dot{\xi}_\nu^\alpha - B(t)\dot{\xi}_0^\alpha + \Big(\frac{\partial R_\nu}{\partial t}(t)\dot{a}^\nu(t) + \frac{\partial R_{\nu\tau}}{\partial t}(t+\tau)\dot{a}_\tau^\nu(t+\tau) - \frac{\partial B}{\partial t}(t)\Big)\xi_0^\alpha + \Big(\frac{\partial R_\nu}{\partial a^\mu}(t)\dot{a}^\nu(t) - \frac{\partial B}{\partial a^\mu}(t)\Big)\xi_\mu^\alpha + \Big(\frac{\partial R_{\nu\tau}}{\partial a_\tau^\mu}(t+\tau)\dot{a}_\tau^\nu(t+\tau) - \frac{\partial B}{\partial a_\tau^\mu}(t+\tau)\Big)\xi_\mu^\alpha = -\dot{G}^\alpha$$
$$t \in [t_1, t_2-\tau]$$
$$R_\nu(t)\dot{\xi}_\nu^\alpha - B(t)\dot{\xi}_0^\alpha + \Big(\frac{\partial R_\nu}{\partial t}(t)\dot{a}^\nu(t) - \frac{\partial B}{\partial t}(t)\Big)\xi_0^\alpha + \Big(\frac{\partial R_\nu}{\partial a^\mu}(t)\dot{a}^\nu(t) - \frac{\partial B}{\partial a^\mu}(t)\Big)\xi_\mu^\alpha = -\dot{G}^\alpha$$
$$t \in (t_2-\tau, t_2] \tag{32}$$

and the conditions(31), the systems (24),(27) have the conserved quantities of Eq. (23).

Proof According to the conditions (31), (32) and noticing Eq. (27), we can derive the conserved quantities of Eq. (23).

Theorem 3 can be called the generalized Noether theorem of the constrained Birkhoffian system with time delay. In addition, if the system is not subject to the constraints, Theorem 2 is reduced to the Noether theorem of free Birkhoffian systems with time delay[22].

6 Example

Consider a fourth-order Birkhoffian system with time delay which describes the motion of a particle with unit mass, and the Birkhoffian and Birkhoff's functions are

$$B = \frac{1}{2}[(a^3(t))^2 + (a^4(t))^2 + (a_\tau^3(t))^2 + (a_\tau^4(t))^2]$$
$$R_1 = a^3(t), R_2 = a^4(t), R_3 = R_4 = 0$$
$$R_{1\tau} = a_\tau^3(t), R_{2\tau} = a_\tau^4(t), R_{3\tau} = R_{4\tau} = 0 \tag{33}$$

and the constraintis

$$f = a^3(t) + bta^4(t) - ba^2(t) + t = 0 \quad b = \text{const} \tag{34}$$

where the Birkhoffian denotes the total energy of the system and formula (34) is linear rheonomic-nonholonomic constraint[18]. The Noether symmetries and conserved quantities of the system are studied.

Eq. (27) gives that

$$-2\dot{a}^3(t) - 2\dot{a}_\tau^3(t+\tau) = 0$$
$$-2\dot{a}^4(t) - 2\dot{a}_\tau^4(t+\tau) = -b\lambda$$
$$\dot{a}^1(t) - a^3(t) + \dot{a}_\tau^1(t+\tau) - a_\tau^3(t+\tau) = \lambda$$
$$\dot{a}^2(t) - a^4(t) + \dot{a}_\tau^2(t+\tau) - a_\tau^4(t+\tau) = bt\lambda$$
$$t \in [t_1, t_2-\tau]$$

$$-2\dot{a}^3(t)=0,\ -2\dot{a}^4(t)=-b\lambda,\ \dot{a}^1(t)-a^3(t)=\lambda$$
$$\dot{a}^2(t)-a^4(t)=bt\lambda \quad t\in(t_2-\tau,t_2] \tag{35}$$

From Eqs. (34), (35), we can find that

$$\lambda=\frac{4}{b^2t} \quad t\in[t_1,t_2-\tau]$$
$$\lambda=\frac{2}{b^2t} \quad t\in(t_2-\tau,t_2] \tag{36}$$

Combining the generalized Birkhoff's Eq. (12) with time delay, we obtain the additional items

$$\Lambda'_1=0,\Lambda'_2=\frac{4}{bt},\Lambda'_3=-\frac{4}{b^2t},\Lambda'_4=-\frac{4}{b}$$
$$t\in[t_1,t_2-\tau]$$
$$\Lambda'_1=0,\Lambda'_2=\frac{2}{bt},\Lambda'_3=-\frac{2}{b^2t},\Lambda'_4=-\frac{2}{b}$$
$$t\in(t_2-\tau,t_2] \tag{37}$$

Next, we study the Noether symmetries and conserved quantities of the corresponding free Birkhoffian system with time delay. The conditions (22) give that

$$a^3(t)\dot{\xi}_1+a^4(t)\dot{\xi}_2+a^3_\tau(t+\tau)\dot{\xi}_1+a^4_\tau(t+\tau)\dot{\xi}_2-B(t)\dot{\xi}_0+\frac{4}{bt}\xi_2+\left(-\frac{4}{bt}\dot{a}^2(t)+\frac{4}{b^2t}\dot{a}^3(t)+\frac{4}{b}\dot{a}^4(t)\right)\xi_0+\left(\dot{a}^1(t)-a^3(t)+\dot{a}^1_\tau(t+\tau)-a^3_\tau(t+\tau)-\frac{4}{b^2t}\right)\xi_3+\left(\dot{a}^2(t)-a^4(t)+\dot{a}^2_\tau(t+\tau)-a^4_\tau(t+\tau)-\frac{4}{b}\right)\xi_4=-\dot{G}(t) \quad t\in[t_1,t_2-\tau]$$
$$a^3(t)\dot{\xi}_1+a^4(t)\dot{\xi}_2-B(t)\dot{\xi}_0+\left(-\frac{2}{bt}\dot{a}^2(t)+\frac{2}{b^2t}\dot{a}^3(t)+\frac{2}{b}\dot{a}^4(t)\right)\xi_0+\frac{2}{bt}\xi_2+\left(\dot{a}^1(t)-a^3(t)-\frac{2}{b^2t}\right)\xi_3+\left(\dot{a}^2(t)-a^4(t)-\frac{2}{b}\right)\xi_4=-\dot{G}(t) \quad t\in(t_2-\tau,t_2] \tag{38}$$

They have the following solutions

$$\xi^1_0=0,\xi^1_1=a^3,\xi^1_2=\xi^1_3=\xi^1_4=0$$
$$G^1=-\frac{1}{2}(a^3(t))^2-\frac{1}{2}(a^3_\tau(t+\tau))^2$$
$$t\in[t_1,t_2-\tau]$$
$$G^1=-\frac{1}{2}(a^3(t))^2$$
$$t\in(t_2-\tau,t_2] \tag{39}$$

and

$$\xi^2_0=0,\xi^2_1=0,\xi_2=1,\xi^2_3=\xi^2_4=0$$
$$G^2=-\frac{4}{b}\ln t \quad t\in[t_1,t_2-\tau]$$
$$G^2=-\frac{2}{b}\ln t \quad t\in(t_2-\tau,t_2] \tag{40}$$

and

$$\xi^3_0=0,\xi^3_1=a^3,\xi^3_2=1,\xi^3_3=\xi^3_4=0$$
$$G^3=-\frac{1}{2}(a^3(t))^2-\frac{1}{2}(a^3_\tau(t+\tau))^2-\frac{4}{b}\ln t \quad t\in[t_1,t_2-\tau]$$
$$G^3=-\frac{1}{2}(a^3(t))^2-\frac{2}{b}\ln t$$
$$t\in(t_2-\tau,t_2] \tag{41}$$

and

$$\xi^4_0=0,\xi^4_1=t,\xi^4_2=1,\xi^4_3=1,\xi^4_4=0$$
$$G^4=\frac{4-4b}{b^2}\ln t-a^1(t)-a^1_\tau(t+\tau)$$
$$t\in[t_1,t_2-\tau]$$
$$G^4=\frac{2-2b}{b^2}\ln t-a^1(t) \quad t\in(t_2-\tau,t_2] \tag{42}$$

and

$$\xi^5_0=0,\xi^5_1=t,\xi^5_2=0,\xi^5_3=1,\xi^5_4=0,$$
$$G^5=\frac{4}{b^2}\ln t-a^1(t)-a^1_\tau(t+\tau) \quad t\in[t_1,t_2-\tau]$$
$$G^5=\frac{2}{b^2}\ln t-a^1(t) \quad t\in(t_2-\tau,t_2] \tag{43}$$

and

$$\xi^6_0=0,\xi^6_1=0,\xi^6_2=t,\xi^6_3=0,\xi^6_4=1$$
$$G^6=-a^2(t)-a^2_\tau(t+\tau) \quad t\in[t_1,t_2-\tau]$$
$$G^6=-a^2(t) \quad t\in(t_2-\tau,t_2] \tag{44}$$

and

$$\xi^7_0=0,\xi^7_1=t,\xi^7_2=t,\xi^7_3=1,\xi^7_4=1$$
$$G^7=\frac{4}{b^2}\ln t-a^1(t)-a^1_\tau(t+\tau)-a^2(t)-a^2_\tau(t+\tau) \quad t\in[t_1,t_2-\tau]$$
$$G^7=\frac{2}{b^2}\ln t-a^1(t)-a^2(t) \quad t\in(t_2-\tau,t_2] \tag{45}$$

Eqs. (39)—(45) correspond to the generalized quasi-symmetric transformations of the system. Theorem 2 gives the conserved quantities as follows

$$I^1=\frac{1}{2}(a^3(t))^2+\frac{1}{2}(a^3_\tau(t+\tau))^2=\text{const}$$
$$t\in[t_1,t_2-\tau]$$
$$I^1=\frac{1}{2}(a^3(t))^2=\text{const} \quad t\in(t_2-\tau,t_2] \tag{46}$$

$$I^2 = a^4(t) + a^4_\tau(t+\tau) - \frac{4}{b}\ln t = \text{const} \quad t \in [t_1, t_2-\tau]$$
$$I^2 = a^4(t) - \frac{2}{b}\ln t = \text{const} \quad t \in (t_2-\tau, t_2] \tag{47}$$

$$I^3 = \frac{1}{2}(a^3(t))^2 + \frac{1}{2}(a^3_\tau(t+\tau))^2 + a^4(t) + a^4_\tau(t+\tau) - \frac{4}{b}\ln t = \text{const} \quad t \in [t_1, t_2-\tau]$$
$$I^3 = \frac{1}{2}(a^3(t))^2 + a^4(t) - \frac{2}{b}\ln t = \text{const} \quad t \in (t_2-\tau, t_2] \tag{48}$$

$$I^4 = a^3(t)t + a^4(t) + a^3_\tau(t+\tau)t + a^4_\tau(t+\tau) + \frac{4-4b}{b^2}\ln t - a^1(t) - a^1_\tau(t+\tau) \quad t \in [t_1, t_2-\tau]$$
$$I^4 = a^3(t)t + a^4(t) + \frac{2-2b}{b^2}\ln t - a^1(t) \quad t \in (t_2-\tau, t_2] \tag{49}$$

$$I^5 = a^3(t)t + a^3_\tau(t+\tau) + \frac{4}{b^2}\ln t - a^1(t) - a^1_\tau(t+\tau) \quad t \in [t_1, t_2-\tau]$$
$$I^5 = a^3(t)t + \frac{2}{b^2}\ln t - a^1(t) \quad t \in (t_2-\tau, t_2] \tag{50}$$

$$I^6 = a^4(t)t + a^4_\tau(t+\tau)t - a^2(t) - a^2_\tau(t+\tau) \quad t \in [t_1, t_2-\tau]$$
$$I^6 = a^4(t)t - a^2(t) \quad t \in (t_2-\tau, t_2] \tag{51}$$

and

$$I^7 = a^3(t)t + a^4(t)t + a^3_\tau(t+\tau)t + a^4_\tau(t+\tau)t + \frac{4}{b^2}\ln t - a^1(t) - a^1_\tau(t+\tau) - a^2(t) - a^2_\tau(t+\tau) \quad t \in [t_1, t_2-\tau]$$
$$I^7 = a^3(t)t + a^4(t)t + \frac{2}{b^2}\ln t - a^1(t) - a^2(t) \quad t \in (t_2-\tau, t_2] \tag{52}$$

Only three of Eqs. (46)—(52) are independent. Actually, we have

$$I^3 = I^1 + I^2 \quad t \in [t_1, t_2-\tau]$$
$$I^3 = I^1 + I^2 \quad t \in (t_2-\tau, t_2] \tag{53}$$
$$I^7 = I^5 + I^6 \quad t \in [t_1, t_2-\tau]$$
$$I^7 = I^5 + I^6 \quad t \in (t_2-\tau, t_2] \tag{54}$$

Then, we study the Noether symmetries and conserved quantities of the constraint Birkhoffian system with time delay. The conditions (31) give

$$-b(\xi_2^\alpha - \dot{a}^2\xi_0^\alpha) + (\xi_3^\alpha - \dot{a}^3\xi_0^\alpha) + bt(\xi_4^\alpha - \dot{a}^4\xi_0^\alpha) = 0 \tag{55}$$

Note that, Eqs. (39), (44) satisfy Eq. (55), therefore Eqs. (39), (44) correspond to the quasi-symmetric transformations of the constrained Birkhoffian system with time delay. And Eqs. (46), (51) are the conserved quantities of the system (Eqs. (33), (34)).

If $b=1$, Eq. (42) satisfies Eq. (55) too, therefore, Eq. (42) also corresponds to the quasi-symmetric transformations. And Eq. (49) is the conserved quantities of the system (Eqs. (33), (34)).

7 Conclusions

The Noether symmetries are studied, as well as the conserved quantities of generalized Birkhoffian systems with time delay. We established the generalized Pfaff-Birkhoff principle (1) with time delay and obtained the generalized Birkhoff's equations (12) with time delay. We discussed the relationship between the symmetries and conserved quantities, and thus, we established the Noether theorem for generalized Birkhoffian systems with time delay. Moreover, we discussed the Noether theory of constrained Birkhoffian systems with time delay. The methods and results of this paper are universal: If there is no time delay, generalized Birkhoffian systems with time delay are reduced to standard generalized Birkhoffian systems. Theorem 1 in the perspective of time delay is reduced to the Noether theorem of standard generalized Birkhoffian systems. It is worth noting that time-delay phenomenon of a system can be connected with a system under the fractional models because of the same characteristic of memory. And many more research fields with obvious time-delay phenomenon are also worth studying.

Acknowledgements

This work was supported by the National Natural Science Foundations of China (Nos. 11572212, 11272227).

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Ms. **Zhai Xianghua** received her bachelor's degree and the master's degree from Suzhou University of Science and Technology in 2011 and 2015, respectively. She is currently pursuing the doctor's degree at Nanjing University of Science and Technology. Her main research area is analytical mechanics.

Prof. **Zhang Yi** received his bachelor's degree and the master's degree from Southeast University in 1983 and 1988, respectively. And in 1999, he received his doctor's degree from Beijing Institute of Technology. Now he works in Suzhou University of Science and Technology, at the same time, he is a doctoral supervisor at Nanjing University of Science and Technology. He is mainly engaged in teaching and scientific research in the field of dynamics and control as well as applied mathematics.

(Production Editor: Zhang Huangqun)