# Two Classes of Quaternary Codes from $\mathbb{Z}_4$-valued Quadratic Forms

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**Abstract**: Let $R=GR(4,m)$ be a Galois ring with Teichmuller set $T_m$ and $Tr_m$ be the trace function from $R$ to $\mathbb{Z}_4$. In this paper, two classes of quaternary codes $C_1=\{\boldsymbol{c}(a,b): a\in R, b\in T_{\frac{m}{2}}\}$, where $\boldsymbol{c}(a,b)=(Tr_m(ax)+Tr_{\frac{m}{2}}(2bx^{2^{\frac{m}{2}}+1}))_{x\in T_m}$, and $C_2=\{\boldsymbol{c}(a,b): a\in R, b\in T_m\}$, where $\boldsymbol{c}(a,b)=(Tr_m(ax+2bx^{2^k+1}))_{x\in T_m}$, and $\frac{m}{\gcd(m,k)}$ is even, are investigated, respectively. The Lee weight distributions, Hamming weight distributions and complete weight distributions of the codes are completely given.

**Key words**: Galois ring; quaternary code; Hamming weight; quadratic form; Lee weight

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## 0 Introduction

Codes over rings were first introduced by Nechaev[1] and later Hammons[2] at the beginning of the 1990's. Since then, codes over rings have been widely investigated[3-10].

Now we recall the definition of quaternary code over $\mathbb{Z}_4$.

**Definition 0.1** If $C$ is an additive subgroup of $\mathbb{Z}_4^n$, we shall call $C$ a quaternary code.

Let $C$ be a quaternary code of length $n$, and let $A_i$ be the number of codewords of weight $i$. Then $A(z):=\sum_{i=0}^{n}A_iz^i$ is called the weight enumerator of $C$. The sequence $(A_i)_{i=0}^n$ is called the weight distribution of $C$.

Hamming distance and Lee distance are two natural metrics for measuring errors for quaternary codes. Then we call $(A_i)_{i=0}^n$ Lee weight distribution with respect to Lee metric and Hamming weight distribution with respect to Hamming metric.

For a vector $\boldsymbol{c}=(c_1,c_2,\cdots,c_n)\in\mathbb{Z}_4^n$, let $N_j$, $j=0,1,2,3$, denote the number of components of $\boldsymbol{c}$ that are equal to $j$. Clearly, we have $N_0+N_1+N_2+N_3=n$. The composition of the vector $\boldsymbol{c}$ is defined to be $\mathrm{comp}(\boldsymbol{c})=(N_0,N_1,N_2,N_3)$.

**Definition 0.2** Let $C$ be a quaternary code over $\mathbb{Z}_4$ and let $A(N_0,N_1,N_2,N_3)$ be the number of codewords $\boldsymbol{c}\in C$ with $\mathrm{comp}(\boldsymbol{c})=(N_0,N_1,N_2,N_3)$. Then the complete weight enumerator of $C$ is the polynomial

$$W_C(z_0,z_1,z_2,z_3)=\sum_{(c_1,c_2,\cdots,c_n)\in C}\left(\prod_{i=1}^{n}z_{c_i}\right)=\sum_{(N_0,N_1,N_2,N_3)\in B}A(N_0,N_1,N_2,N_3)z_0^{N_0}z_1^{N_1}z_2^{N_2}z_3^{N_3}$$

where $B=\{\mathrm{comp}(\boldsymbol{c}): \boldsymbol{c}\in C\}$, the sequence $(A(N_0,N_1,N_2,N_3))_{(N_0,N_1,N_2,N_3)\in B}$ is called the complete weight distribution of $C$. As far as we know, there are few papers which gave the complete weight enumerators of linear codes over rings.

Recently, some quaternary codes are constructed derived from $\mathbb{Z}_4$-valued quadratic forms. In Ref. [11], $\mathbb{Z}_4$-valued quadratic form was introduced by Brown for the first time. Then

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Schmidt[12-13] gave the basic theories of $\mathbb{Z}_4$-valued quadratic forms. He also constructed a class of quaternary sequences based on $\mathbb{Z}_4$-valued quadratic forms. Let $m$, $k$ be positive integers with $\gcd(m,k)=d$, a class of an interesting exponential sum over Galois ring is

$$\rho(a,b)=\sum_{x\in T_m}(\sqrt{-1})^{Tr_m(ax+2bx^{2^k+1})}$$
$$a\in R, b\in T_m$$

where $R=GR(4,m)$ is a Galois ring with Teichmuller set $T_m$ and $Tr_m$ is the trace function from $R$ to $\mathbb{Z}_4$. This class of exponential sum is related to a $\mathbb{Z}_4$-valued quadratic for $Q(x)=Tr_m(ax+2bx^{2^k+1})$. We have some well-known results about $\rho(a,b)$ as follows:

(1) For $k=1$, the distribution[6] of $\rho(a,b)$ was studied to determine the correlation distribution of a quaternary sequence family.

(2) For odd $m$ and $\gcd(m,k)=1$, codes with the same weight distribution as the Goethals codes and Delsarte-Goethals codes[14] were obtained based on $\rho(a,b)$.

(3) For odd $m$ and $k=1$, the theory of $\mathbb{Z}_4$-valued quadratic forms[15] was used to analyze the exponential sum $\rho(a,b)$ and new sequence families were obtained.

(4) For $\gcd(m,k)=d$ and $\frac{m}{d}$ being odd, several classes of codes and sequences derived from $\rho(a,b)$ were constructed[3].

However, for the case that $\frac{m}{\gcd(m,k)}$ is even, the distribution of $\rho(a,b)$ was not given.

In this paper, two classes of exponential sums $\rho_1(a,b)=\sum_{x\in T_m}(\sqrt{-1})^{Tr_m(ax)+Tr_{\frac{m}{2}}(2bx^{2^{\frac{m}{2}}+1})}$, $a\in R$, $b\in T_{\frac{m}{2}}$, where $m$ is an even integer, and $\rho_2(a,b)=\sum_{x\in T_m}(\sqrt{-1})^{Tr_m(ax+2bx^{2^k+1})}$, $a\in R, b\in T_m$, where $\frac{m}{\gcd(m,k)}$ is even, are investigated, respectively. Through the discussions on the solutions of certain equations derived from $\mathbb{Z}_4$-valued quadratic forms, the distributions of the exponential sums are completely determined. We investigate two classes of quaternary codes based on the exponential sums $C_1=\{\boldsymbol{c}(a,b): a\in R, b\in T_{\frac{m}{2}}\}$, where $\boldsymbol{c}(a,b)=(Tr_m(ax)+Tr_{\frac{m}{2}}(2bx^{2^{\frac{m}{2}}+1}))_{x\in T_m}$ and $C_2=\{\boldsymbol{c}(a,b): a\in R, b\in T_m\}$, where $\boldsymbol{c}(a,b)=(Tr_m(ax+2bx^{2^k+1}))_{x\in T_m}$ and $\frac{m}{\gcd(m,k)}$ is even. The Lee weight distributions, Hamming weight distributions and complete weight distributions of the codes are given. Moreover, the complete weight distributions of the codes in Ref. [3] can also be determined by the method given in this paper.

## 1 Preliminaries

Throughout this paper, we adopt the following notation unless otherwise stated.

$\mathbb{Z}_n$ the residue class ring modulo $n$;

$\mathbb{F}_q$ the finite field with $q$ elements;

$R_m$ the Galois ring $GR(4,m)$;

$T_m$ the Teichmuller representative set of $R_m$;

$tr_m$ the trace function from $T_m$ to $\mathbb{Z}_2$;

$Tr_m$ the trace function from $R_m$ to $\mathbb{Z}_4$;

$tr_{m/\frac{m}{2}}$ the trace function from $T_m$ to $T_{\frac{m}{2}}$;

$\mathfrak{Re}(x)$ the real part of $x$;

$\mathfrak{Im}(x)$ the imaginary part of $x$.

### 1.1 Galois rings

Some preliminaries about Galois rings are given below.

For positive integers $m\geqslant 1$, let $\mathbb{Z}_4$ be the ring of integers modulo 4 and $f$ a monic basic irreducible polynomial of degree $m$ in $\mathbb{Z}_4[x]$. The ring $R_m=\mathbb{Z}_4[x]/(f)$ is called the Galois ring, which is a Galois extension of the ring $\mathbb{Z}_4$ and denoted by $R_m=GR(4,m)$, where $R_m$ is a finite chain ring of length 3 and its unique maximal ideal is $2R_m$, i. e. $\{0\}=4R_m<2R_m<R_m$.

The group of units of the Galois ring contains a unique cyclic multiplicative group $T_m^*$ of order $2^m-1$. If $\xi_m$ is a generator of this group, then $T_m^*=<\xi_m>$ and the set $T_m=T_m^*\cup\{0\}=\{0,1,\xi_m,\cdots,\xi_m^{2^m-2}\}$ is called the Teichmuller representative set of $R_m$.

For every $z\in R_m$, it can be uniquely expressed in the form of $z=x+2y$, $x, y\in T_m$. Clearly, the addition operation in the Teichmuller set $T_m$ is not closed. Specially, for arbitrary $x$, $y\in T_m$, there exists a unique $z\in T_m$ such that $z=x+y+2\sqrt{xy}$. For convenience, a new operation $\oplus$ on $T_m$ is defined as $x\oplus y=x+y+2\sqrt{xy}$.

Let $\mathbb{F}_{2^m}$ denote the Galois field $GF(2^m)$, then $(T_m,\oplus,\cdot)\cong\mathbb{F}_{2^m}$.

The Frobenius automorphism $\sigma$ on $T_m$ is given by $\sigma(x)=x^2$. The trace functions $tr_m: T_m\rightarrow\mathbb{Z}_2$ and $Tr_m: R_m\rightarrow\mathbb{Z}_4$ are defined as

$$tr_m(x)=\bigoplus_{j=0}^{m-1}\sigma^j(x)=\bigoplus_{j=0}^{m-1}x^{2^j} \qquad x\in T_m$$

$$Tr_m(x+2y)=\sum_{j=0}^{m-1}(x^{2^j}+2y^{2^j}) \qquad x,y\in T_m$$

One can easily check that $tr_m(\sigma(x))=tr_m(x)$ and $2Tr_m(x)=2tr_m(x)$ hold for all $x\in T_m$.

For more information about Galois rings, readers can refer to Refs. [3,13].

One can easily deduce the following lemma.

**Lemma 1.1** Let $tr$ denote the trace function from $\mathbb{F}_{2^m}$ to $\mathbb{F}_2$ and $Tr_m$ denote the trace function from $R_m$ to $\mathbb{Z}_4$. Let $\mu$ be the reduction modulo 2. Then Fig. 1 shows the communities.

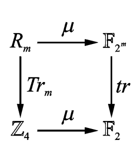


Fig. 1 Schematic diagram of trace function

The proof is obvious, so we omit it here.

## 1.2 $\mathbb{Z}_4$-valued quadratic forms

In the following, we present some results about $\mathbb{Z}_4$-valued quadratic forms [11,13,16].

Let $K:=\{z\in\mathbb{Z}_4: z^2=z\}$ be the Teichmuller representatives in $\mathbb{Z}_4$. Informally, we identify $K$ as $\mathbb{Z}_2=\{0,1\}$, which is a subset of $\mathbb{Z}_4$, in this paper.

**Definition 1.2** A symmetric bilinear form on $T_m$ is a mapping $B: T_m\times T_m\rightarrow K$ with two properties:

(1) Symmetry: $B(x,y)=B(y,x)$;

(2) Bilinearity: for any $\alpha,\beta\in K$, $B(\alpha x\oplus\beta y, z)=\alpha B(x,z)\oplus\beta B(y,z)$.

We call $B$ alternating if $B(x,x)=0$ for all $x\in T_m$. Otherwise it is called nonalternating. One can see that $T_m$ is an $m$-dimensional vector space over $\mathbb{Z}_2$. Let $\{\lambda_0,\lambda_1,\cdots,\lambda_{m-1}\}$ be a basis for $T_m$ over $\mathbb{Z}_2$. Then, relative to this basis, $B$ is uniquely determined by its matrix of size $m\times m$ given by $\boldsymbol{B}=(b_{jk})_{0\leqslant j,k<m}$, where $b_{jk}=B(\lambda_j,\lambda_k)$. Let $\boldsymbol{x}=(x_0,x_1,\cdots,x_{m-1})$ and $\boldsymbol{y}=(y_0,y_1,\cdots,y_{m-1})$ be the $\mathbb{Z}_2$-valued coordinate vectors of $x$ and $y$ relative to the basis $\{\lambda_0,\lambda_1,\cdots,\lambda_{m-1}\}$, respectively. Hence $x=\bigoplus_{j=0}^{m-1}\lambda_j x_j$ and $y=\bigoplus_{j=0}^{m-1}\lambda_j y_j$. Then it is easy to verify that $B(x,y)=\boldsymbol{x}\boldsymbol{B}\boldsymbol{y}^{\mathrm{T}}$.

**Definition 1.3** A $\mathbb{Z}_4$-valued quadratic form on $T_m$ is a mapping $\boldsymbol{Q}$[11]: $T_m\rightarrow\mathbb{Z}_4$ with two properties:

(1) $\boldsymbol{Q}(0)=0$;

(2) $\boldsymbol{Q}(x\oplus y)=\boldsymbol{Q}(x)+\boldsymbol{Q}(y)+2B(x,y)$

where $B$ is a symmetric bilinear form defined as above. $\boldsymbol{Q}$ is called alternating if its associated bilinear form $B$ is alternating. The rank of $B$ is defined as

$$\mathrm{rank}(B)=m-\dim_{\mathbb{Z}_2}(rad(B))$$

where $rad(B)=\{x\in F: B(x,y)=0 \text{ for all } y\in T_m\}$. The rank of $\boldsymbol{Q}$ is defined as $\mathrm{rank}(\boldsymbol{Q})=\mathrm{rank}(B)$.

Now let $\boldsymbol{x}=(x_0,x_1,\cdots,x_{m-1})$ be the $\mathbb{Z}_2$-valued coordinate vector of $\boldsymbol{x}$ relative to the basis $\{\lambda_0,\lambda_1,\cdots,\lambda_{m-1}\}$ for $T_m$ over $\mathbb{Z}_2$. Hence $x=\bigoplus_{j=0}^{m-1}\lambda_j x_j$. And from Definitions 1.2 and 1.3, $\boldsymbol{Q}(x)=\boldsymbol{Q}(\bigoplus_{j=0}^{m-1}\lambda_j x_j)=\sum_{j=0}^{m-1}x_j\boldsymbol{Q}(\lambda_j)+2\sum_{0\leqslant j<k<m}x_j x_k B(\lambda_j,\lambda_k)=\sum_{0\leqslant j,k<m}x_j x_k B(\lambda_j,\lambda_k)+\sum_{j=0}^{m-1}x_j(\boldsymbol{Q}(\lambda_j)-B(\lambda_i,\lambda_j))$. $\boldsymbol{Q}(x)-B(x,x)\in 2\mathbb{Z}_2$ can be easilydeduced from $2\boldsymbol{Q}(x)=2B(x,x)$. Therefore, there exists $\boldsymbol{v}\in\mathbb{Z}_2^m$ such that $\boldsymbol{Q}(x)=\boldsymbol{x}\boldsymbol{B}\boldsymbol{x}+2\boldsymbol{v}\,\boldsymbol{x}^{\mathrm{T}}$.

It is known[12] that every $\mathbb{Z}_4$-valued quadratic form can be written uniquely in the following form.

**Lemma 1.4** Every $\mathbb{Z}_4$-valued quadratic form $\boldsymbol{Q}: T_m\rightarrow\mathbb{Z}_4$ can be written uniquely as

$$\boldsymbol{Q}(x)=Tr_m(a_0x)+2\sum_{j=1}^{\lfloor\frac{m}{2}\rfloor}Tr_{s_{2^j+1}}(a_jx^{2^j+1})$$

where $a_j \in T_{s_{2^j+1}}$ and $s_{2^j+1}$ is the size of the cyclotomic coset having coset leader $2^j+1$. Furthermore, if $m \geqslant 3$, $s_{2^j+1}=m$ except for $s_{2^{\frac{m}{2}}+1}=\frac{m}{2}$ when $m$ is even.

For a $\mathbb{Z}_4$-valued quadratic form $Q: T_m \to \mathbb{Z}_4$, the distribution of the values of an interesting exponential sum $\chi_Q(\lambda)=\sum_{x\in T_m}(\sqrt{-1})^{Q(x)}(-1)^{tr(\lambda x)}$ was investigated[13,17] as follows, where $\lambda$ ranges over $T_m$.

**Lemma 1.5** If $Q$ is an alternating $\mathbb{Z}_4$-valued quadratic form of rank $r$, then the distribution of $\{\chi_Q(\lambda), \lambda \in T_m\}$ is given in Table 1.

**Table 1 Value distribution of $\{\chi_Q(\lambda), \lambda\in T_m\}$ for alternating $Q$**

| Value | Frequency |
|---|---|
| $0$ | $2^m-2^r$ |
| $\pm 2^{m-\frac{r}{2}}$ | $2^{r-1}\pm 2^{\frac{r}{2}-1}$ |

**Lemma 1.6** If $Q$ is a nonalternating $\mathbb{Z}_4$-valued quadratic form of rank $r$, then the distributions of $\{\chi_Q(\lambda), \lambda\in T_m\}$ are given in Tables 2 , 3, respectively.

**Table 2 Value distribution of $\{\chi_Q(\lambda), \lambda\in T_m\}$ for odd $r$**

| Value | Frequency |
|---|---|
| $0$ | $2^m-2^r$ |
| $\pm(1+\sqrt{-1})2^{m-\frac{r+1}{2}}$ | $2^{r-2}\pm 2^{\frac{r-3}{2}}$ |
| $\pm(1+\sqrt{-1})2^{m-\frac{r+1}{2}}$ | $2^{r-2}\pm 2^{\frac{r-3}{2}}$ |

**Table 3 Value distribution of $\{\chi_Q(\lambda), \lambda\in T_m\}$ for even $r$**

| Value | Frequency |
|---|---|
| $0$ | $2^m-2^r$ |
| $\pm 2^{m-\frac{r}{2}}$ | $2^{r-2}\pm 2^{\frac{r}{2}-1}$ |
| $\pm 2^{m-\frac{r}{2}}\sqrt{-1}$ | $2^{r-2}$ (each) |

## 2 Codes Derived from the First $\mathbb{Z}_4$-valued Quadratic Form

Throughout this section, let $m$ be an even integer. In this section, a class of quaternary codes derived from a $\mathbb{Z}_4$-valued quadratic forms is investigated. The Lee weight distributions, Hamming weight distributions and complete weight distributions of the quaternary codes are given, respectively.

### 2.1 Distribution of the first exponential sums $\rho_1(a,b)$

A class of exponential sum over the Galois ring $R$ is denoted by

$$\rho_1(a,b)=\sum_{x\in T_m}(\sqrt{-1})^{Tr_m(ax)+Tr_{\frac{m}{2}}(2bx^{2^{\frac{m}{2}}+1})}$$
$$a\in R, b\in T_{\frac{m}{2}}$$

Let $a=c+2c'$, where $c,c'\in T_m$. Hence

$$\rho_1(a,b)=\xi(b,c,c')=$$
$$\sum_{x\in T_m}(\sqrt{-1})^{Tr_m((c+2c')x)+Tr_{\frac{m}{2}}(2bx^{2^{\frac{m}{2}}+1})}=$$
$$\sum_{x\in T_m}(\sqrt{-1})^{Tr_m(cx)+Tr_{\frac{m}{2}}(2bx^{2^{\frac{m}{2}}+1})}(-1)^{tr_m(c'x)}=$$
$$\sum_{x\in T_m}(\sqrt{-1})^{Q_{b,c}(x)}(-1)^{tr_m(c'x)}$$

where $Q_{b,c}(x)=Tr_m(cx)+Tr_{\frac{m}{2}}^{(2bx^{2^{\frac{m}{2}}+1})}$ is a $\mathbb{Z}_4$-valued quadratic form from Lemma 1.4.

To determine the distribution of the exponential sum $\rho_1(a,b)$, it is sufficient to consider the rank distribution of the quadratic form $Q_{b,c}(x)=Tr_m(cx)+Tr_{\frac{m}{2}}(2bx^{2^{\frac{m}{2}}+1})$, since

$$Q_{b,c}(x\oplus y)=$$
$$Tr_m(c(x\oplus y))+Tr_{\frac{m}{2}}(2b(x\oplus y)^{2^{\frac{m}{2}}+1})=$$
$$Tr_m(cx+cy+2c\sqrt{xy})+$$
$$Tr_{\frac{m}{2}}(2bx^{2^{\frac{m}{2}}+1}\oplus 2by^{2^{\frac{m}{2}}+1}\oplus 2bxy^{2^{\frac{m}{2}}}\oplus 2bx^{2^{\frac{m}{2}}}y)=$$
$$Tr_m(cx+cy)+2tr_m(c^2xy)+$$
$$Tr_{\frac{m}{2}}(2bx^{2^{\frac{m}{2}}+1}+2by^{2^{\frac{m}{2}}+1})+2tr_{\frac{m}{2}}(bxy^{2^{\frac{m}{2}}}\oplus bx^{2^{\frac{m}{2}}}y))$$

We have the associated bilinear form

$$B_{b,c}(x,y)=tr_m(c^2xy)\oplus tr_{\frac{m}{2}}(bxy^{2^{\frac{m}{2}}}\oplus bx^{2^{\frac{m}{2}}}y)$$

We need the following lemma later.

**Lemma 2.1** Let $tr_{m/\frac{m}{2}}(x)=x\oplus x^{2^{\frac{m}{2}}}$ be the trace function from $T_m$ to $T_{\frac{m}{2}}$. Then for every $b\in T_{\frac{m}{2}}$, there exists $d\in T_m$ such that $b=d\oplus d^{2^{\frac{m}{2}}}$.

**Proof** The proof is completed by noting that $tr_{m/\frac{m}{2}}: T_m\to T_{\frac{m}{2}}$ is a surjection.

By Lemma 2.1 and some techniques, we have the following results.

**Lemma 2.2** The bilinear form (2.1) equals to $tr_m(c^{2^{\frac{m}{2}+1}}x^{2^{\frac{m}{2}}}y^{2^{\frac{m}{2}}}\oplus bxy^{2^{\frac{m}{2}}})$.

**Proof** From Lemma 2.1, we know that $b=d\oplus d^{2^{\frac{m}{2}}}$ for $d\in T_m$. Thus

$$tr_m(dxy^{2^{\frac{m}{2}}}\oplus dx^{2^{\frac{m}{2}}}y)=$$
$$tr_{\frac{m}{2}}tr_{m/\frac{m}{2}}(dxy^{2^{\frac{m}{2}}}\oplus dx^{2^{\frac{m}{2}}}y)=$$
$$tr_{\frac{m}{2}}(dxy^{2^{\frac{m}{2}}}\oplus dx^{2^{\frac{m}{2}}}y\oplus d^{2^{\frac{m}{2}}}x^{2^{\frac{m}{2}}}y\oplus d^{2^{\frac{m}{2}}}xy^{2^{\frac{m}{2}}}=$$
$$tr_{\frac{m}{2}}((d\oplus d^{2^{\frac{m}{2}}})(xy^{2^{\frac{m}{2}}}\oplus x^{2^{\frac{m}{2}}}y))=$$
$$tr_{\frac{m}{2}}(b(xy^{2^{\frac{m}{2}}}\oplus x^{2^{\frac{m}{2}}}y))$$

It follows $B_{b,c}(x,y)=tr_m(c^2xy\oplus dxy^{2^{\frac{m}{2}}}\oplus dx^{2^{\frac{m}{2}}}y)=tr_m(c^{2^{\frac{m}{2}+1}}x^{2^{\frac{m}{2}}}y^{2^{\frac{m}{2}}})\oplus tr_m(dxy^{2^{\frac{m}{2}}})\oplus tr_m(d^{2^{\frac{m}{2}}}xy^{2^{\frac{m}{2}}})=tr_m(c^{2^{\frac{m}{2}+1}}x^{2^{\frac{m}{2}}}y^{2^{\frac{m}{2}}}\oplus bxy^{2^{\frac{m}{2}}})$, which completes the proof.

By Lemma 2.2, in order to determine the rank of $B_{b,c}(x,y)$, it is sufficient to consider the roots of the equation $c^{2^{\frac{m}{2}+1}}x^{2^{\frac{m}{2}}}\oplus bx=0$. Let $g(x)=c^{2^{\frac{m}{2}+1}}x^{2^{\frac{m}{2}}-1}\oplus b$, it becomes $xg(x)=0$.

**Lemma 2.3** Let $T_m^*=\langle\xi\rangle$. For $b\in T_{\frac{m}{2}}^*$, $c\in T_m^*$, the equation $x^{2^{\frac{m}{2}}-1}=\frac{b}{c^2}$ has $2^{\frac{m}{2}}-1$ solutions in and no solution in $T_m$ if $c^2\in b\langle\xi^{2^{\frac{m}{2}}-1}\rangle$. Let $N_j$ denote the number of $(b,c)\in T_{\frac{m}{2}}^*\times T_m^*$ such that the equation has exactly $j$ roots in $T_m$. Then $N_0=(2^m-1)(2^{\frac{m}{2}}-2)$, $N_{2^{\frac{m}{2}}-1}=2^m-1$.

**Proof** Note that the equation $x^{2^{\frac{m}{2}}-1}=\frac{b}{c^2}$ has $2^{\frac{m}{2}}-1$ solutions or no solution. Let $\frac{b}{c^2}=\xi^t$ for $0\leqslant t<2^m-1$. Assume that $\xi^s$ is a solution of the equation. Hence $s(2^{\frac{m}{2}}-1)\equiv t \pmod{2^m-1}$ which implies that $(2^{\frac{m}{2}}-1)\mid t$. Then $\frac{b}{c^2}\in\langle\xi^{2^{\frac{m}{2}}-1}\rangle$ which implies that $c^2\in b\langle\xi^{2^{\frac{m}{2}}-1}\rangle$. Assume that $T_{\frac{m}{2}}^*=\{b_1,b_2,\cdots,b_{\frac{m}{2}-1}\}$. Note that $T_m^*=b_1\langle\xi^{2^{\frac{m}{2}}-1}\rangle\dot\cup\cdots\dot\cup b_{\frac{m}{2}-1}\langle\xi^{2^{\frac{m}{2}}-1}\rangle$, and the sets on the right are disjoint from each other. Choose a fixed $b$ in $T_{\frac{m}{2}}^*$, then the number of $c$ satisfying $c^2\in b\langle\xi^{2^{\frac{m}{2}}-1}\rangle$ equals to $|b\langle\xi^{2^{\frac{m}{2}}-1}\rangle|$. The proof is completed.

Now we determine the rank distribution of $B_{b,c}(x,y)$. To achieve this goal, we define $R_j=\{(b,c)\in T_{\frac{m}{2}}\times T_m\setminus\{(0,0)\}:\mathrm{rank}(B_{b,c}(x,y))=m-j\}$.

**Theorem 2.4** When $(b,c)$ runs over $T_{\frac{m}{2}}\times T_m\setminus\{(0,0)\}$, the distribution of $\mathrm{rank}(B_{b,c}(x,y))$ is given by

$$\begin{cases}|R_0|=2^m(2^{\frac{m}{2}}-1)\\|R_{\frac{m}{2}}|=2^m-1\end{cases}$$

**Proof** It is sufficient to consider the following cases of the roots of $g(x)$.

**Case 1** When $c=0,b\neq0$, it is clear that $g(x)$ has no solution in $T_m$. In this case, $\mathrm{rank}(B_{b,c}(x,y))=m$.

**Case 2** When $b=0,c\neq0$, it is clear that $g(x)$ has no nonzero solution in $T_m$. In this case, $\mathrm{rank}(B_{b,c}(x,y))=m$.

**Case 3** When $b\neq0,c\neq0$, $g(x)=0$ becomes $(c^2x)2^{\frac{m}{2}}-1=\frac{b}{c^2}$. From Lemma 2.3, this equation has $2^{\frac{m}{2}}-1$ solutions in $T_m$ if $c^2\in b\langle\xi^{2^{\frac{m}{2}}-1}\rangle$ and no solution in $T_m$ if $c^2\in b\langle\xi^{2^{\frac{m}{2}}-1}\rangle$. In this case, $\mathrm{rank}(B_{b,c}(x,y))=\frac{m}{2}$ or $\mathrm{rank}(B_{b,c}(x,y))=m$.

Then the conclusion follows.

From the discussions above, we get the distribution of the exponential sums $\rho_1(a,b)$ in the following.

**Theorem 2.5** When $(a,b)$ runs through $R\times T_{\frac{m}{2}}$, the distribution of the exponential sums $\rho_1(a,b)$ is given as follows.

If $m\equiv0 \pmod 4$, $\rho_1(a,b)$ has the following distribution

$$\begin{cases}2^m & \text{once}\\0 & (2^m-1)(2^m-2^{\frac{m}{2}}+1)\text{times}\\\pm2^{\frac{m}{2}} & (2^{\frac{m}{2}}-1)(2^{m-1}\pm2^{\frac{m}{2}-1}+\\ & (2^{m-2}\pm2^{\frac{m}{2}-1})(2^m-1))\text{times}\\\pm2^{\frac{m}{2}}\sqrt{-1} & 2^{m-2}(2^m-1)(2^{\frac{m}{2}}-1)\text{times(each)}\\\pm2^{\frac{3m}{4}} & (2^{\frac{m}{2}-2}\pm2^{\frac{m}{4}-1})(2^m-1)\text{times}\\\pm2^{\frac{3m}{4}}\sqrt{-1} & (2^m-1)2^{\frac{m}{2}-2}\text{times(each)}\end{cases}$$

If $m\equiv2 \pmod 4$, $\rho_1(a,b)$ has the following distribution

$$\begin{cases} 2^m & \text{once} \\ 0 & (2^m-1)(2^m-2^{\frac{m}{2}}+1)\text{times} \\ \pm 2^{\frac{m}{2}} & (2^{\frac{m}{2}}-1)(2^{m-1}\pm 2^{\frac{m}{2}-1}+ \\ & (2^{m-2}\pm 2^{\frac{m}{2}-1})(2^m-1))\text{times} \\ \pm 2^{\frac{m}{2}}\sqrt{-1} & 2^{m-2}(2^m-1)(2^{\frac{m}{2}}-1) \\ & \text{times(each)} \\ \pm(1+\sqrt{-1})2^{\frac{3m-2}{4}} & (2^{\frac{m}{2}-2}\pm 2^{\frac{m-6}{4}})(2^m-1)\text{times} \\ \pm(1-\sqrt{-1})2^{\frac{3m-2}{4}} & (2^{\frac{m}{2}-2}\pm 2^{\frac{m-6}{4}})(2^m-1)\text{times} \end{cases}$$

**Proof** Let $a=c+2c'$, where $c,c'\in T_m$. The proof of this theorem is divided into the following cases.

**Case 1** $b=c=0$. This is a trivial case, one can verify that

$$\xi(0,0,c')=\sum_{x\in T_m}(-1)tr_m(c'x)=\begin{cases}0 & c'\neq 0\\ 2^m & c=0\end{cases}$$

**Case 2** $b\neq 0$ and $c=0$. For $c=0$, we can see that $Q_{(b,0)}(x)$ is alternating and $\text{rank}(Q)=m$ where $m$ is even. Hence by Lemma 1.5, we have $\xi(b,0,c')=\pm 2^{\frac{m}{2}}$ which occurs $(2^{\frac{m}{2}}-1)(2^{m-1}\pm 2^{\frac{m}{2}-1})$ times when $(b,c')$ runs through $T^*_{\frac{m}{2}}\times T_m$.

**Case 3** $b=0$ and $c\neq 0$. In this case, $Q_{(0,c)}(x)$ is nonalternating and $\text{rank}(Q)=m$. Hence by Lemma 1.6,

$$\xi(0,c,c')=\begin{cases}\pm 2^{\frac{m}{2}} & (2^{m-2}\pm 2^{\frac{m}{2}-1})(2^m-1)\text{times}\\ \pm 2^{\frac{m}{2}}\sqrt{-1} & 2^{m-2}(2^m-1)\text{times(each)}\end{cases}$$

when $(c,c')$ runs through $T^*_m\times T_m$.

**Case 4** $b\neq 0$ and $c\neq 0$. In this case, $Q_{(b,c)}(x)$ is nonalternating. For $bc\neq 0$, $\text{rank}(Q)=m$ or $\frac{m}{2}$. Note that $\frac{m}{2}$ is even if $m\equiv 0 \pmod 4$ and odd if $m\equiv 2 \pmod 4$. By Theorem 2.4 and Lemma 1.6, when $(b,c)$ runs through $R_0\backslash\{(0,c),(b,0): b\in T^*_{\frac{m}{2}}, c\in T^*_m\}$ and $c'$ runs through $T_m$, we have

$$\xi(b,c,c')=$$

$$\begin{cases}\pm 2^{\frac{m}{2}} & (2^{m-2}\pm 2^{\frac{m}{2}-1})(2^m-1)(2^{\frac{m}{2}}-2)\text{times}\\ \pm 2^{\frac{m}{2}}\sqrt{-1} & 2^{m-2}(2^m-1)(2^{\frac{m}{2}}-2)\text{times(each)}\end{cases}$$

When $(b,c)$ runs through $R_{\frac{m}{2}}$, $c'$ runs through $T_m$ and $m\equiv 0 \pmod 4$, one can deduce that

$$\xi(b,c,c')=$$

$$\begin{cases}0 & (2^m-2^{\frac{m}{2}})(2^m-1)\text{times}\\ \pm 2^{\frac{3m}{4}} & (2^{\frac{m}{2}-2}\pm 2^{\frac{m}{4}-1})(2^m-1)\text{times}\\ \pm 2^{\frac{3m}{4}}\sqrt{-1} & 2^{\frac{m}{2}-2}(2^m-1)\text{times(each)}\end{cases}$$

from Theorem 2.4 and Lemma 1.6. When $(b,c)$ runs through $R_{\frac{m}{2}}$, $c'$ runs through $T_m$ and $m\equiv 2 \pmod 4$, one can deduce that

$$\xi(b,c,c')=$$

$$\begin{cases}0 & (2^m-2^{\frac{m}{2}})(2^m-1)\text{times}\\ \pm(1+\sqrt{-1})2^{\frac{3m-2}{4}} & \\ & (2^{\frac{m}{2}-2}\pm 2^{\frac{m-6}{4}})(2^m-1)\text{times}\\ \pm(1-\sqrt{-1})2^{\frac{3m-2}{4}} & \\ & (2^{\frac{m}{2}-2}\pm 2^{\frac{m-6}{4}})(2^m-1)\text{times}\end{cases}$$

from Theorem 2.4 and Lemma 1.6.

Then the conclusion follows.

### 2.2 Lee weight distribution and Hamming weight distribution of $C_1$

In this subsection, we investigate the Lee weight distributions and Hamming weight distributions of several classes of codes from $\rho_1(a,b)$, respectively.

For an element $z\in\mathbb{Z}_4$, define its Lee weight as $w_L(z)=1-\Re e((\sqrt{-1})^z)$ where $\Re e(t)$ denotes the real part of a complex number $t$. For a codeword $\boldsymbol{c}=(c_1,c_2,\cdots,c_n)$, the Lee weight of $\boldsymbol{c}$ is defined as

$$w_L(\boldsymbol{c})=n-\Re e(\sum_{j=1}^{n}(\sqrt{-1})c_j)$$

Now we define a quaternary code as

$$C_1=\{\boldsymbol{c}(a,b):\ a\in R, b\in T_{\frac{m}{2}}\}$$

where $\boldsymbol{c}(a,b)=(Tr_m(ax)+Tr_{\frac{m}{2}}(2bx^{2^{\frac{m}{2}}+1}))_{x\in T_m}$. Due to the definition of Lee weight, we have

$$w_L(\boldsymbol{c}(a,b))=$$

$$2^m-\Re e(\sum_{x\in T_m}(\sqrt{-1})^{Tr_m(ax)+Tr_{\frac{m}{2}}(2bx^{2^{\frac{m}{2}+1}})})$$

Then from Theorem 2.5, we can determine the Lee weight distribution of $C_1$ in the following.

**Theorem 2.6** When $(a,b)$ runs through $R\times T_{\frac{m}{2}}$, the Lee weight distributions of the quaternary code $C_1$ are given in Tables 4, 5.

**Table 4 Value distribution of Lee weight with $m\equiv 0 \pmod 4$**

| Weight | Frequency |
|---|---|
| 0 | 1 |
| $2^m$ | $(2^m-1)\cdot(2^{\frac{3m}{2}-1}-2^{\frac{m}{2}-1}+2^{m-1}+1)$ |
| $2^m\mp 2^{\frac{m}{2}}$ | $(2^{\frac{m}{2}}-1)(2^{m-1}\pm 2^{\frac{m}{2}-1}+(2^{m-2}\pm 2^{\frac{m}{2}-1})(2^m-1))$ |
| $2^m\mp 2^{\frac{3m}{4}}$ | $(2^{\frac{m}{2}-2}\pm 2^{\frac{m}{4}-1})(2^m-1)$ |

**Table 5 Value distribution of Lee weight with $m\equiv 2 \pmod 4$**

| Weight | Frequency |
|---|---|
| $0$ | $1$ |
| $2^m$ | $(2^m-1)\cdot(2^{\frac{3m}{2}-1}-2^{\frac{m}{2}-1}+2^{m-1}+1)$ |
| $2^m\mp 2^{\frac{m}{2}}$ | $(2^{\frac{m}{2}}-1)(2^{m-1}\pm 2^{\frac{m}{2}-1}+(2^{m-2}\pm 2^{\frac{m}{2}-1})(2^m-1))$ |
| $2^m\mp 2^{\frac{3m-2}{4}}$ | $(2^{\frac{m}{2}-2}\pm 2^{\frac{m}{4}-1})(2^m-1)$ |

Define the Hamming weight of a codeword $\boldsymbol{c}=(c_1,c_2,\cdots,c_n)$ as $w_{\mathrm{H}}(\boldsymbol{c})=\#\{1\leqslant j\leqslant n: c_j\neq 0\}$. Now we determine the Hamming weight of the quaternary code $C_1$. For a codeword $\boldsymbol{c}(a,b)\in C$, the Hamming weight of it can be expressed as

$$w_{\mathrm{H}}(\boldsymbol{c}(a,b))=2^m-|\{x\in T_m:\boldsymbol{c}(a,b)=0\}|=$$

$$2^m-\frac{1}{4}\sum_{x\in T_m}\sum_{\lambda\in\mathbb{Z}_4}(\sqrt{-1})\lambda\boldsymbol{c}(a,b)=$$

$$2^m-\frac{1}{4}\sum_{\lambda\in\mathbb{Z}_4}\sum_{x\in T_m}(\sqrt{-1})^{Tr_m(\lambda ax)+Tr_{\frac{m}{2}}(2\lambda bx^{2^{\frac{m}{2}}+1})}=$$

$$2^m-\frac{1}{4}\sum_{\lambda\in\mathbb{Z}_4}\rho_1(\lambda a,\lambda b)=$$

$$2^m-\frac{1}{4}\cdot 2^m-\frac{1}{4}\rho_1(a,b)-\frac{1}{4}\rho_1(2a,2b)-$$

$$\frac{1}{4}\rho_1(3a,3b)=3\cdot 2^{m-2}-\frac{1}{4}\rho_1(a,b)-$$

$$\frac{1}{4}\rho_1(2a,2b)-\frac{1}{4}\overline{\rho_1(a,b)}=$$

$$3\cdot 2^{m-2}-\frac{1}{2}\Re\mathrm{e}(\rho_1(a,b))-\frac{1}{4}\rho_1(2a,2b)$$

Assume that $a=c+2c',c,c'\in T_m$. It is easy to verify that $\rho_1(2a,2b)=2^m$ if $c=0$ and $\rho_1(2a,2b)=0$ otherwise. Thus

$$w_{\mathrm{H}}(\boldsymbol{c}(a,b))=$$

$$\begin{cases}2^{m-1}-\frac{1}{2}\Re\mathrm{e}(\rho_1(a,b)) & c=0\\ 3\cdot 2^{m-2}-\frac{1}{2}\Re\mathrm{e}(\rho_1(a,b)) & \text{otherwise}\end{cases}$$

From the proof of Theorem 2.5, it is easy to get the distribution of the exponential sums $\xi(b,0,c')$ as

$$\xi(b,0,c')=\begin{cases}0 & (2^m-1)\text{ times}\\ 2^m & \text{once}\\ \pm 2^{\frac{m}{2}} & (2^{\frac{m}{2}}-1)(2^{m-1}\pm 2^{\frac{m}{2}-1})\text{times}\end{cases}$$

For $c\neq 0$ and $m\equiv 0 \pmod 4$, one can obtain the distribution of the exponential sums $\xi(b,c,c')$ as

$$\xi(b,c,c')=$$

$$\begin{cases}0 & (2^m-2^{\frac{m}{2}})(2^m-1)\text{times}\\ \pm 2^{\frac{m}{2}} & (2^{m-2}\pm 2^{\frac{m}{2}-1})(2^m-1)(2^{\frac{m}{2}}-1)\text{times}\\ \pm 2^{\frac{m}{2}}\sqrt{-1} & 2^{m-2}(2^m-1)(2^{\frac{m}{2}}-1)\text{times(each)}\\ \pm 2^{\frac{3m}{4}} & (2^{\frac{m}{2}-2}\pm 2^{\frac{m}{4}-1})(2^m-1)\text{times}\\ \pm 2^{\frac{3m}{4}}\sqrt{-1} & 2^{\frac{m}{2}-2}(2^m-1)\text{times(each)}\end{cases}$$

For $c\neq 0$ and $m\equiv 2 \pmod 4$, one can obtain the distribution of the exponential sums $\xi(b,c,c')$ as

$$\xi(b,c,c')=$$

$$\begin{cases}0 & (2^m-2^{\frac{m}{2}})(2^m-1)\text{times}\\ \pm 2^{\frac{m}{2}} & (2^{m-2}\pm 2^{\frac{m}{2}-1})(2^m-1)\cdot(2^{\frac{m}{2}}-1)\text{times}\\ \pm 2^{\frac{m}{2}}\sqrt{-1} & 2^{m-2}(2^m-1)(2^{\frac{m}{2}}-1)\text{ times(each)}\\ \pm(1+\sqrt{-1})2^{\frac{3m-2}{4}} & (2^{\frac{m}{2}-2}\pm 2^{\frac{m-6}{4}})(2^m-1)\text{times}\\ \pm(1-\sqrt{-1})2^{\frac{3m-2}{4}} & (2^{\frac{m}{2}-2}\pm 2^{\frac{m-6}{4}})(2^m-1)\text{times}\end{cases}$$

Therefore, the Hamming weight distribution of the quaternary codes is obtained.

**Theorem 2.7** When $(a,b)$ runs through $R\times T_{\frac{m}{2}}$, the Hamming weight distributions of $C_1$ are given in Tables 6, 7.

**Table 6 Value distribution of Hamming weight with $m\equiv 0 \pmod 4$**

| Weight | Frequency |
|---|---|
| $0$ | $1$ |
| $2^{m-1}$ | $2^m-1$ |
| $3\cdot 2^{m-2}$ | $(2^m-1)\cdot(2^{\frac{3m}{2}-1}+2^{m-1}-2^{\frac{m}{2}-1})$ |
| $2^{m-1}\mp 2^{\frac{m}{2}-1}$ | $(2^{\frac{m}{2}}-1)(2^{m-1}\pm 2^{\frac{m}{2}-1})$ |
| $3\cdot 2^{m-2}\mp 2^{\frac{m}{2}-1}$ | $(2^m-1)(2^{m-2}\pm 2^{\frac{m}{2}-1})\cdot(2^{\frac{m}{2}}-1)$ |
| $3\cdot 2^{m-2}\mp 2^{\frac{3m}{4}-1}$ | $(2^m-1)(2^{\frac{m}{2}-2}\pm 2^{\frac{m}{4}-1})$ |

**Table 7 Value distribution of Hamming weight with $m\equiv 2 \pmod 4$**

| Weight | Frequency |
|---|---|
| $0$ | $1$ |
| $2^{m-1}$ | $2^m-1$ |
| $3\cdot 2^{m-2}$ | $(2^m-1)\cdot(2^{\frac{3m}{2}-1}+2^{m-1}-2^{\frac{m}{2}})$ |
| $2^{m-1}\mp 2^{\frac{m}{2}-1}$ | $(2^{\frac{m}{2}}-1)(2^{m-1}\pm 2^{\frac{m}{2}-1})$ |
| $3\cdot 2^{m-2}\mp 2^{\frac{m}{2}-1}$ | $(2^{m-2}\pm 2^{\frac{m}{2}-1})\cdot(2^m-1)(2^{\frac{m}{2}}-1)$ |
| $3\cdot 2^{m-2}\mp 2^{\frac{3m-2}{4}-1}$ | $(2^m-1)\cdot(2^{\frac{m}{2}-1}\pm 2^{\frac{m-2}{4}})$ |

**Remark 2.8** Let $\varphi$ be the Gray map over $\mathbb{Z}_4$, define

$$\varphi(C_1)=\{\varphi(\boldsymbol{c}(a,b)):a\in R,b\in T_{\frac{m}{2}}\}$$

where $\boldsymbol{c}(a,b)$ is defined as above. From Ref. [17], $w_{\mathrm{H}}(\varphi(\boldsymbol{c}(a,b)))=w_{\mathrm{L}}(\boldsymbol{c}(a,b))=2^m-\Re\mathrm{e}(\rho_1(a,b))$. Thus the Hamming weight distribution of $\varphi(C_1)$ is the same as the Lee weight distribution of $C_1$.

### 2.3 Complete weight distribution of $C_1$

In this subsection, we investigate the complete weight distribution of the quaternary code $C_1$. Let $N_i=N_{\boldsymbol{c}(a,b)}$, $i=0,1,2,3$ denote the number of components of $\boldsymbol{c}(a,b)$ that are equal to $i$.

Define a function on $\mathbb{Z}_4$ as $f(z)=1-\Im\mathrm{m}((\sqrt{-1})^z)$, $z\in\mathbb{Z}_4$.

Then for a codeword $\boldsymbol{c}=(c_1,c_2,\cdots,c_n)$ over $\mathbb{Z}_4^n$, $f(\boldsymbol{c})=n-\Im\mathrm{m}(\sum_{j=1}^{n}(\sqrt{-1})c_j)$.

Consider the system of equations

$$\begin{cases}f(\boldsymbol{c}(a,b))=N_0+N_2+2N_3\\ w_{\mathrm{H}}(\boldsymbol{c}(a,b))=N_1+N_2+N_3\\ w_{\mathrm{L}}(\boldsymbol{c}(a,b))=N_1+2N_2+N_3\\ 2^m=N_0+N_1+N_2+N_3\end{cases}$$

One can deduce that

$$N_0=2^m-w_{\mathrm{H}}(\boldsymbol{c}(a,b))=\begin{cases}2^{m-1}+\frac{1}{2}\Re\mathrm{e}(\rho_1(a,b)) & c=0\\ 2^{m-2}+\frac{1}{2}\Re\mathrm{e}(\rho_1(a,b)) & c\neq 0\end{cases}$$

$$N_1=\begin{cases}0 & c=0\\ 2^{m-2}+\frac{1}{2}\Im\mathrm{m}(\rho_1(a,b)) & c\neq 0\end{cases}$$

$$N_2=w_{\mathrm{L}}(\boldsymbol{c}(a,b))-w_{\mathrm{H}}(\boldsymbol{c}(a,b))=\begin{cases}2^{m-1}-\frac{1}{2}\Re\mathrm{e}(\rho_1(a,b)) & c=0\\ 2^{m-2}-\frac{1}{2}\Re\mathrm{e}(\rho_1(a,b)) & c\neq 0\end{cases}$$

$$N_3=\begin{cases}0 & c=0\\ 2^{m-2}-\frac{1}{2}\Im\mathrm{m}(\rho_1(a,b)) & c\neq 0\end{cases}$$

In the following, we give the distributions of $(N_0,N_1,N_2,N_3)$ when $(a,b)$ runs through $R\times T_{\frac{m}{2}}$.

**Theorem 2.9** The complete weight enumerator of the quaternary code $C_1$ is given in Table 8 if $m\equiv 0 \pmod 4$ and Table 9 if $m\equiv 2 \pmod 4$ when $(a,b)$ runs through $R\times T_{\frac{m}{2}}$.

**Table 8 Complete weight enumerator of $C_1$ with $m\equiv 0 \pmod 4$**

| $N_0$ | $N_1$ | $N_2$ | $N_3$ | Frequency |
|---|---|---|---|---|
| $2^{m-1}$ | 0 | $2^{m-1}$ | 0 | $2^m-1$ |
| $2^m$ | 0 | 0 | 0 | 1 |
| $2^{m-1}\pm 2^{\frac{m}{2}-1}$ | 0 | $2^{m-1}\mp 2^{\frac{m}{2}-1}$ | 0 | $(2^{\frac{m}{2}}-1)(2^{m-1}\pm 2^{\frac{m}{2}-1})$ |
| $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}$ | $(2^m-2^{\frac{m}{2}})(2^m-1)$ |
| $2^{m-2}\pm 2^{\frac{m}{2}-1}$ | $2^{m-2}$ | $2^{m-2}\mp 2^{\frac{m}{2}-1}$ | $2^{m-2}$ | $(2^{m-2}\pm 2^{\frac{m}{2}-1})(2^m-1)(2^{\frac{m}{2}}-1)$ |
| $2^{m-2}$ | $2^{m-2}\pm 2^{\frac{m}{2}-1}$ | $2^{m-2}$ | $2^{m-2}\mp 2^{\frac{m}{2}-1}$ | $2^{m-2}(2^m-1)(2^{\frac{m}{2}}-1)$ (each) |
| $2^{m-2}\pm 2^{\frac{3m}{4}-1}$ | $2^{m-2}$ | $2^{m-2}\mp 2^{\frac{3m}{4}-1}$ | $2^{m-2}$ | $(2^{\frac{m}{2}-2}\pm 2^{\frac{m}{4}-1})(2^m-1)$ |
| $2^{m-2}$ | $2^{m-2}\pm 2^{\frac{3m}{4}-1}$ | $2^{m-2}$ | $2^{m-2}\mp 2^{\frac{3m}{4}-1}$ | $2^{\frac{m}{2}-2}(2^m-1)$ (each) |

**Table 9 Complete weight enumerator of $C_1$ with $m\equiv 2 \pmod 4$**

| $N_0$ | $N_1$ | $N_2$ | $N_3$ | Frequency |
|---|---|---|---|---|
| $2^{m-1}$ | 0 | $2^{m-1}$ | 0 | $2^m-1$ |
| $2^m$ | 0 | 0 | 0 | 1 |
| $2^{m-1}\pm 2^{\frac{m}{2}-1}$ | 0 | $2^{m-1}\mp 2^{\frac{m}{2}-1}$ | 0 | $(2^{\frac{m}{2}}-1)(2^{m-1}\pm 2^{\frac{m}{2}-1})$ |
| $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}$ | $(2^m-2^{\frac{m}{2}})(2^m-1)$ |
| $2^{m-2}\pm 2^{\frac{m}{2}-1}$ | $2^{m-2}$ | $2^{m-2}\mp 2^{\frac{m}{2}-1}$ | $2^{m-2}$ | $(2^{m-2}\pm 2^{\frac{m}{2}-1})(2^m-1)(2^{\frac{m}{2}}-1)$ |
| $2^{m-2}$ | $2^{m-2}\pm 2^{\frac{m}{2}-1}$ | $2^{m-2}$ | $2^{m-2}\mp 2^{\frac{m}{2}-1}$ | $2^{m-2}(2^m-1)(2^{\frac{m}{2}}-1)$ (each) |
| $2^{m-2}\pm 2^{\frac{3m-6}{4}}$ | $2^{m-2}\pm 2^{\frac{3m-6}{4}}$ | $2^{m-2}\mp 2^{\frac{3m-6}{4}}$ | $2^{m-2}\mp 2^{\frac{3m-6}{4}}$ | $(2^{\frac{m}{2}-2}\pm 2^{\frac{m-6}{4}})(2^m-1)$ |
| $2^{m-2}\pm 2^{\frac{3m-6}{4}}$ | $2^{m-2}\mp 2^{\frac{3m-6}{4}}$ | $2^{m-2}\mp 2^{\frac{3m-6}{4}}$ | $2^{m-2}\pm 2^{\frac{3m-6}{4}}$ | $(2^{\frac{m}{2}-2}\pm 2^{\frac{m-6}{4}})(2^m-1)$ |

**Proof** We only give the proof for $m\equiv 0\pmod 4$ since the case for $m\equiv 2\pmod 4$ can be similarly proved. The proof is given in several cases.

**Case 1** $c=0$ and $\rho_1(a,b)=\xi(b,0,c')=0$. In this case, $N_0=2^{m-1}$, $N_2=2^m-2^{m-1}=2^{m-1}$, $N_1=N_3=0$. From Theorem 2.5, $(N_0,N_1,N_2,N_3)=(2^{m-1},0,2^{m-1},0)$ occurs $2^m-1$ times.

**Case 2** $c=0$ and $\rho_1(a,b)=\xi(b,0,c')=2^m$. In this case, $N_0=2^{m-1}+\frac{1}{2}\Re(\rho_1(a,b))=2^m$, $N_1=N_2=N_3=0$. From Theorem 2.5, $(N_0,N_1,N_2,N_3)=(2^m,0,0,0)$ occurs one time.

**Case 3** $c=0$ and $\rho_1(a,b)=\xi(b,0,c')=\pm 2^{\frac{m}{2}}$. In this case, $N_0=2^{m-1}\pm 2^{\frac{m}{2}-1}$, $N_1=N_3=0$, $N_2=2^{m-1}\mp 2^{\frac{m}{2}-1}$. From Theorem 2.5, $(N_0,N_1,N_2,N_3)=(2^{m-1}\pm 2^{\frac{m}{2}-1},0,2^{m-1}\mp 2^{\frac{m}{2}-1},0)$ occurs $(2^{\frac{m}{2}}-1)(2^{m-1}\pm 2^{\frac{m}{2}-1})$ times.

**Case 4** $c\neq 0$ and $\rho_1(a,b)=\xi(b,c,c')=0$. In this case, $N_0=N_1=N_2=N_3=2^{m-2}$. From Theorem 2.5, $(N_0,N_1,N_2,N_3)=(2^{m-2},2^{m-2},2^{m-2},2^{m-2})$ occurs $(2^m-2^{\frac{m}{2}})(2^m-1)$ times.

**Case 5** $c\neq 0$ and $\rho_1(a,b)=\xi(b,c,c')=\pm 2^{\frac{m}{2}}$. In this case, $N_0=2^{m-2}\pm 2^{\frac{m}{2}-1}$, $N_1=N_3=2^{m-2}$, $N_2=2^{m-2}-\frac{1}{2}\Re(\rho_1(a,b))=2^{m-2}\mp 2^{\frac{m}{2}-1}$. From Theorem 2.5, $(N_0,N_1,N_2,N_3)=(2^{m-2}\pm 2^{\frac{m}{2}-1},2^{m-2},2^{m-2}\mp 2^{\frac{m}{2}-1},2^{m-2})$ occurs $(2^{m-2}\pm 2^{\frac{m}{2}-1})(2^m-1)(2^{\frac{m}{2}}-1)$ times.

**Case 6** $c\neq 0$ and $\rho_1(a,b)=\xi(b,c,c')=\pm 2^{\frac{m}{2}}\cdot\sqrt{-1}$. In this case, $N_0=2^{m-2}+\frac{1}{2}\Re(\rho_1(a,b))=2^{m-2}$, $N_1=2^{m-2}+\frac{1}{2}\Im(\rho_1(a,b))=2^{m-2}\pm 2^{\frac{m}{2}-1}$, $N_2=2^{m-2}-\frac{1}{2}\Re(\rho_1(a,b))=2^{m-2}$, $N_3=2^{m-2}-\frac{1}{2}\Im(\rho_1(a,b))=2^{m-2}\mp 2^{\frac{m}{2}-1}$.

From Theorem 2.5, each of $(N_0,N_1,N_2,N_3)=(2^{m-2},2^{m-2}\pm 2^{\frac{m}{2}-1},2^{m-2},2^{m-2}\mp 2^{\frac{m}{2}-1})$ occurs $2^{m-2}(2^m-1)(2^{\frac{m}{2}}-1)$ times.

**Case 7** $c\neq 0$ and $\rho_1(a,b)=\xi(b,c,c')=\pm 2^{\frac{3m}{4}}$. In this case, $N_0=2^{m-2}+\frac{1}{2}\Re(\rho_1(a,b))=2^{m-2}\pm 2^{\frac{3m}{4}-1}$, $N_1=N_3=2^{m-2}$, $N_2=2^{m-2}-\frac{1}{2}\Re(\rho_1(a,b))=2^{m-2}\mp 2^{\frac{3m}{4}-1}$. From Theorem 2.5, $(N_0,N_1,N_2,N_3)=(2^{m-2}\pm 2^{\frac{3m}{4}-1},2^{m-2},2^{m-2}\mp 2^{\frac{3m}{4}-1},2^{m-2})$ occurs $(2^{\frac{m}{2}-2}\pm 2^{\frac{m}{4}-1})(2^m-1)$ times.

**Case 8** $c\neq 0$ and $\rho_1(a,b)=\xi(b,c,c')=\pm 2^{\frac{3m}{4}}\cdot\sqrt{-1}$. In this case, $N_0=2^{m-2}+\frac{1}{2}\Re(\rho_1(a,b))=2^{m-2}$, $N_1=2^{m-2}+\frac{1}{2}\Im(\rho_1(a,b))=2^{m-2}\pm 2^{\frac{3m}{4}-1}$, $N_2=2^{m-2}-\frac{1}{2}\Re(\rho_1(a,b))=2^{m-2}$, $N_3=2^{m-2}-\frac{1}{2}\Im(\rho_1(a,b))=2^{m-2}\mp 2^{\frac{3m}{4}-1}$. From Theorem 2.5, each of $(N_0,N_1,N_2,N_3)=(2^{m-2},2^{m-2}\pm 2^{\frac{3m}{4}-1},2^{m-2},2^{m-2}\mp 2^{\frac{3m}{4}-1})$ occurs $2^{\frac{m}{2}-2}(2^m-1)$ times.

Then the proof is completed by combining Cases 1—8.

**Remark 2.10** Quaternary codes derived from a class of exponential sums

$$\rho(a,b)=\sum_{x\in T_m}(\sqrt{-1})\,Tr_m(ax+2bx^{2^k+1})$$
$$a\in R, b\in T_m$$

where $\gcd(m,k)=k$ and $\frac{m}{k}$ being odd, were investigated[3]. The Hamming weight distributions and the Lee weight distributions of the codes were given. However, the complete weight distributions were not investigated. Using the same techniques given by us, the complete weight distributions of the quaternary codes[3] can be easily determined. We omit them here.

## 3 Codes Derived from the Second Quadratic Form

Throughout this section, we always assume that $\gcd(m,k)=d$ and $\frac{m}{d}$ is even. In this section, we investigate quaternary codes derived from a class of exponential sum $\rho_2(a,b)=\sum_{x\in T_m}(\sqrt{-1})^{Tr_m(ax+2bx^{2^k+1})}$, where $a=c+2c'\in R$,

$b,c,c'\in T_m$. Hence

$$\rho_2(a,b)=\xi_2(b,c,c')=$$
$$\sum_{x\in T_m}(\sqrt{-1})^{Tr_m((c+2c')x+2bx^{2^k+1})}=$$
$$\sum_{x\in T_m}(\sqrt{-1})^{Tr_m(cx+2bx^{2^k+1})}(-1)^{tr_m(c'x)}=$$
$$\sum_{x\in T_m}(\sqrt{-1})^{Q_{b,c}(x)}(-1)^{tr_m(c'x)}$$

where $Q_{b,c}(x)=Tr_m(cx+2bx^{2^k+1})$. For the case $\frac{m}{\gcd(m,k)}$ odd, the reader is referred to Ref. [3].

### 3.1 Distribution of exponential sums $\rho_2(a,b)$

To determine the distribution of the exponential sum $\rho_2(a,b)$, it is sufficient to consider the rank distribution of the quadratic form $Q_{b,c}(x)=Tr_m(cx+2bx^{2^k+1}),b,c,x\in T_m$.

Note that $Q_{b,c}(x)$ is a $\mathbb{Z}_4$-valued quadratic form with associated bilinear form

$$B_{b,c}(x,y)=tr_m(c^2xy\oplus bx^{2^k}y\oplus bxy^{2^k})=$$
$$tr_m(c^{2^{k+1}}x^{2^k}y^{2^k})\oplus tr_m(b^{2^k}x^{2^{2k}}y^{2^k})\oplus tr_m(bxy^{2^k})=$$
$$tr_m((c^{2^{k+1}}x^{2^k}\oplus b^{2^k}x^{2^{2k}}\oplus bx)y^{2^k})$$

Hence it is sufficient to consider the roots of the equation

$$c^{2^{k+1}}x^{2^k}\oplus b^{2^k}x^{2^{2k}}\oplus bx=0 \qquad (1)$$

In the following, we discuss the roots of Eq. (1) in several cases.

**Lemma 3.1** Let $T_m^*=\langle\xi\rangle$. If $b=0,c\neq0$, the unique solution of Eq. (1) is $x=0$ in $T_m$ and $\mathrm{rank}(B_{b,c}(x,y))=m$. If $b\in\langle\xi^{2^d+1}\rangle,c=0$, then Eq. (1) has $2^{2d}$ solutions in $T_m$ and $\mathrm{rank}(B_{b,c}(x,y))=m-2d$. If $b\in\langle\xi^{2^d+1}\rangle,c=0,b\neq0$, then Eq. (1) has the unique solution $x=0$ in $T_m$ and $\mathrm{rank}(B_{b,c}(x,y))=m$.

**Proof** If $b=0,c\neq0$, the proof is obvious.

Now we consider the case $b\neq0,c=0$. In this case, Eq. (1) becomes $b^{2^k}x^{2^{2k}}\oplus bx=0$. Hence it is sufficient to consider the roots of $b^{2^k}x^{2^{2k}-1}\oplus b=0$ which implies that $x^{2^{2k}-1}=b^{1-2^k}$. Since $\gcd(2k,m)=d\times\gcd(2\frac{k}{d},\frac{m}{d})=2d$, one can deduce that $\gcd(2^{2k}-1,2^m-1)=2^{\gcd(2k,m)}-1=2^{2d}-1$. Hence $x^{2^{2k}-1}=b^{1-2^k}$ has no solution or $2^{2d}-1$ solutions in $T_m$. Assume that $x=\xi^s$ is a solution of it and $b=\xi^t$, then we have $\xi^{s(2^{2k}-1)}=\xi^{t(1-2^k)}$ which implies that $s(2^{2k}-1)\equiv t(1-2^k) \pmod{2^m-1}$. Then it is equivalent to $\gcd(2^{2k}-1,2^m-1)\mid t(1-2^k)$, i.e. $2^{2d}-1\mid t(1-2^k)$. Since $\gcd(2^m-1,2^k-1)=2^d-1$ and $\gcd(2^m-1,2^k+1)=2^d+1$, we have $2^d+1\mid t$. Then $x^{2^{2k}-1}=b^{1-2^k}$ has solutions in $T_m$ if and only if $b\in\langle\xi^{2^d+1}\rangle$. The proof is completed.

To consider the case $bc\neq0$, we need the following lemmas.

**Lemma 3.2**[18] Let $h(x)=x^{2^k+1}\oplus\epsilon x\oplus\epsilon$ with $\epsilon\in T_m^*$ and $D=2^{\gcd(k,m)}$. Then $h(x)=0$ has 0, 1, 2 or $D+1$ solutions in $T_m$. Assume that $n_j:=\#\{\epsilon\in T_m* \mid h(x)=0 \text{ has } j \text{ roots in } T_m\}$.

If $\nu=\frac{m}{\gcd(m,k)}$ is even, then $n_0=\frac{D^{\nu+1}-D}{2(D+1)}$, $n_1=D^{\nu-1}$, $n_2=\frac{(D-2)(D^\nu-1)}{2(D-1)}$, $n_{D+1}=\frac{D^{\nu-1}-D}{D^2-1}$. Furthermore, if $h(x)=0$ has a unique solution $x_0\in T_m$, $(x_0\oplus1)^{\frac{2^m-1}{D-1}}=1$.

Let $g(y)=c^{2^{k+1}}y\oplus b^{2^k}y^{1+2^k}\oplus b$, where $y=x^{2^k-1}$. Then Eq. (1) is equivalent to $xg(y)=0$. If $bc\neq0$, rewrite $g(y)$ as $g(y)=b\gamma^{-1}(z^{2^k+1}\oplus\gamma z\oplus\gamma)$, where $\gamma=\frac{c^{2^{k+1}(2^k+1)}}{b^{2^{k+1}}}$ and $y=\frac{b}{c^{2^{k+1}}}z$. For each fixed $c\in T_m^*$, $\gamma$ runs through $T_m^*$ when $b$ runs through $T_m^*$. We define a map $\sigma$ as follows

$$T_m^*\times T_m^*\rightarrow T_m^*$$
$$(b,c)\rightarrow\gamma$$

Note that this map is asubjective homomorphism of two groups. Since $|\ker(\sigma)|=2^m-1$, there exist $2^m-1$ pairs of $(b,c)$ such that $\sigma(b,c)=\gamma$ for each $\gamma\in T_m^*$. From Lemma 3.2, we know that $g(y)$ has 0, 1, 2, or $1+2^d$ solutions in $T_m$.

**Lemma 3.3** For $b\neq0$, let

$$g(y)=c^{2^{k+1}}y\oplus b^{2^k}y^{1+2^k}\oplus b$$

where $y=x^{2^k-1}$. Then:

(1) If $y_1$ and $y_2$ are two different solutions of $g(y)=0$, $(y_1y_2)^{\frac{2^m-1}{2^d-1}}=1$.

(2) If $g(y)=0$ has at least three solutions $y_1,y_2,y_3\in T_m$, $y_i^{\frac{2^m-1}{2^d-1}}=1$ for $i=1,2,3$.

(3) If $g(y)=0$ has a unique solution $y_0\in$

$T_m$, $y_0^{\frac{2^m-1}{2^d-1}}=1$.

**Proof** (1) Since $y_1y_2(y_1\oplus y_2)2^k=y_1^{2^k+1}y_2\oplus y_2^{2^k+1}y_1=(\frac{b}{c^{2^{k+1}}})2^{k+1}\frac{b}{c^{2^{k+1}}}(z_1z_2^{2^k+1}\oplus z_2z_1^{2^k+1})=(\frac{b}{c^{2^{k+1}}})2^{k+1}\frac{b}{c^{2^{k+1}}}(z_1(\gamma z_2\oplus\gamma)\oplus z_2(\gamma z_1\oplus\gamma))=(\frac{b}{c^{2^{k+1}}})2^{k+1}\frac{b}{c^{2^{k+1}}}\gamma(z_1\oplus z_2)=\frac{b}{c^{2^{k+1}}}(z_1\oplus z_2)$, we get that $y_1y_2=(y_1\oplus y_2)1-2^k$.

Note that $\gcd(2^m-1,2^k-1)=2^d-1$. Then the conclusion follows.

(2) Since $y_1^2=\frac{(y_1y_2)(y_1y_3)}{y_2y_3}$, we have $(y_1^2)^{\frac{2^m-1}{2^d-1}}=1$ from (1). Note that $\gcd(2,2^m-1)=1$, we have $y_1^{\frac{2^m-1}{2^d-1}}=1$. Similarly, we have $y_i^{\frac{2^m-1}{2^d-1}}=1$ for $i=2,3$.

(3) Since $y_0$ is the unique solution of $g(y)=0$, we know that $z_0=\frac{c^{2^{k+1}}}{b}y_0$ is the unique solution of the equation $z^{2^k+1}\oplus\gamma z_0\oplus\gamma=0$. From Lemma 3.2, $1=(z_0\oplus 1)^{\frac{2^m-1}{2^d-1}}=\left(\frac{c^{2^{k+1}}}{b}y_0\oplus 1\right)^{\frac{2^m-1}{2^d-1}}$. Since $b^{2^k}y_0^{1+2^k}\oplus c^{2^{k+1}}y_0\oplus b=0$, we have $\frac{c^{2^{k+1}}}{b}y_0\oplus 1=b^{2^k-1}y_0^{1+2^k}$. Hence $1=(b^{2^k-1}y_0^{1+2^k})^{\frac{2^m-1}{2^d-1}}=y_0^{\frac{(1+2^k)(2^m-1)}{2^d-1}}$. Since $\gcd(1+2^k,2^k-1)=1$ and $2^d-1\,|\,2^k-1$, we have $\gcd(1+2^k,2^d-1)=1$. Then one can deduce that $y_0^{\frac{2^m-1}{2^d-1}}=1$.

Denote $R_j$ as $R_j=\{(b,c)\in T_m\times T_m\backslash\{(0,0\}:\operatorname{rank}(Q_b(x))=m-j\}$.

From Lemma 3.1 and Lemma 3.2, we can get the rank distribution of $B_{b,c}(x,y)$ when $(b,c)$ runs through $T_m\times T_m\backslash\{(0,0\}$.

**Theorem 3.4** When $(b,c)$ runs through $T_m\times T_m\backslash\{(0,0\}$, the distribution of $\operatorname{rank}(B_{b,c}(x,y))$ is

$$\begin{cases}|R_0|=(2^m-1)\cdot\\ \quad\left(\frac{2^{m+d-1}+3\cdot 2^{d-1}+1}{2^d+1}+\frac{(2^{d-1}-1)(2^m-1)}{2^d-1}\right)\\ |R_d|=2^{m-d}(2^m-1)\\ |R_{2d}|=(2^m-1)\frac{2^{m-d}-1}{2^{2d}-1}\end{cases}$$

**Proof** We know that $x^{2^k-1}=y$ has $2^d-1$ solutions if and only if $y$ is a $(2^d-1)$th power in $T_m^*$ and no solution if and only if $y$ is not a $(2^d-1)$th power in $T_m^*$. Now we discuss the rank of $B_{b,c}(x,y)$ in several cases.

**Case 1** $b=0,c\neq 0$. From Lemma 3.1, $\operatorname{rank}(B_{b,c}(x,y))=m$.

**Case 2** $b\in\langle\xi^{2^d+1}\rangle,c=0$. From Lemma 3.1, $\operatorname{rank}(B_{b,c}(x,y))=m-2d$. This case occurs $\frac{2^m-1}{2^d+1}$ times.

**Case 3** $b\in\langle\xi^{2^d+1}\rangle,c=0$. From Lemma 3.1, $\operatorname{rank}(B_{b,c}(x,y))=m$. This case occurs $2^m-1-\frac{2^m-1}{2^d+1}$ times.

**Case 4** $bc\neq 0$ and $g(y)=0$ has no solution. In this case, $\operatorname{rank}(B_{b,c}(x,y))=m$. From Lemma 3.2, This case occurs $(2^m-1)n_0$ times.

**Case 5** $bc\neq 0$ and $g(y)=0$ has exactly one solution. Then from Lemma 3.3, this solution is a $(2^d-1)$th power in $T_m$. This implies that Eq. (1) has $2^d$ solutions. Hence $\operatorname{rank}(B_{b,c}(x,y))=m-d$. From Lemma 3.2, this case occurs $(2^m-1)n_1$ times.

**Case 6** $bc\neq 0$ and $g(y)=0$ has two solutions in $T_m$. Then from Lemma 3.3, these two solutions are both $(2^d-1)$th powers or both not $(2^d-1)$th powers in $T_m$. If the former is true, then Eq. (1) would have $2\times(2^d-1)+1=2^{d+1}-1$ roots in $T_m$, which is impossible. Hence both roots are not $(2^d-1)$th powers and Eq. (1) has the unique zero solution. Then $\operatorname{rank}(B_{b,c}(x,y))=m$ and this case occurs $(2^m-1)n_2$ times.

**Case 7** $bc\neq 0$ and $g(y)=0$ has $2^d+1$ solutions. Then from Lemma 3.3, all of them are $(2^d-1)$th powers in $T_m$. This implies that Eq. (1) has $2^{2d}$ solutions. Hence $\operatorname{rank}(B_{b,c}(x,y))=m-2d$. From Lemma 3.2, This case occurs $(2^m-1)n_{2^d+1}$ times.

The proof is completed by combining Cases 1—7.

Now we determine the distribution of the exponential sum $\rho_2(a,b)$.

**Theorem 3.5** When $(a,b)$ runs through $R\times T_m$, the distribution of the exponential sum $\rho_2(a,b)$ is given as follows.

If $d=\gcd(m,k)$ is odd,

$$\rho_2(a,b)=\begin{cases} 2^m & \text{once} \\ 0 & A_1 \text{ times} \\ \pm 2^{\frac{m}{2}} & A_2 \text{ times} \\ \pm 2^{\frac{m}{2}}\sqrt{-1} & A_3 \text{ times(each)} \\ \pm 2^{m-\frac{m-2d}{2}} & A_4 \text{ times} \\ \pm 2^{m-\frac{m-2d}{2}}\sqrt{-1} & A_5 \text{ times} \\ \pm(1+\sqrt{-1})2^{m-\frac{m-d+1}{2}} & A_6 \text{ times} \\ \pm(1-\sqrt{-1})2^{m-\frac{m-d+1}{2}} & A_7 \text{ times} \end{cases}$$

where

$$\begin{cases} A_1=(2^m-1)(1+(2^d-1)(2^{m-2d}+2^{2m-2d})+2^{m-2d}(2^{m-d}-2^d)) \\ A_2=(2^m-1)\left(\dfrac{2^d}{2^d+1}(2^{m-1}\pm 2^{\frac{m}{2}-1})+\left(1+\dfrac{(2^m-1)(2^{2d}-2^d-1)}{2^{2d}-1}\right)(2^{m-2}\pm 2^{\frac{m}{2}-1})\right) \\ A_3=(2^m-1)2^{m-2}\left(1+(2^m-1)\dfrac{2^{2d}-2^d-1}{2^{2d}-1}\right) \\ A_4=\dfrac{(2^m-1)(2^{m-2d-1}\pm 2^{\frac{m-2d}{2}-1})}{2^d+1}+\dfrac{(2^m-1)(2^{m-d}-2^d)(2^{m-2d-2}\pm 2^{\frac{m-2d}{2}-1})}{2^{2d}-1} \\ A_5=(2^m-1)\dfrac{2^{m-2d-2}(2^{m-d}-2^d)}{2^{2d}-1} \\ A_6=(2^m-1)2^{m-d}(2^{m-d-2}\pm 2^{\frac{m-d-3}{2}}) \\ A_7=(2^m-1)2^{m-d}(2^{m-d-2}\pm 2^{\frac{m-d-3}{2}}) \end{cases}$$

If $d=\gcd(m,k)$ is even

$$\rho_2(a,b)=\begin{cases} 2^m & \text{once} \\ 0 & B_1 \text{ times} \\ \pm 2^{\frac{m}{2}} & B_2 \text{ times} \\ \pm 2^{\frac{m}{2}}\sqrt{-1} & B_3 \text{ times(each)} \\ \pm 2^{m-\frac{m-2d}{2}} & B_4 \text{ times} \\ \pm 2^{m-\frac{m-2d}{2}}\sqrt{-1} & B_5 \text{ times(each)} \\ \pm 2^{m-\frac{m-d}{2}} & B_6 \text{ times} \\ \pm 2^{m-\frac{m-d}{2}}\sqrt{-1} & B_7 \text{ times(each)} \end{cases}$$

where

$$\begin{cases} B_1=(2^m-1)(1+(2^d-1)(2^{m-2d}+2^{2m-2d})+2^{m-2d}(2^{m-d}-2^d)) \\ B_2=(2^m-1)\left(\dfrac{2^d}{2^d+1}(2^{m-1}\pm 2^{\frac{m}{2}-1})+\left(1+\dfrac{(2^m-1)(2^{2d}-2^d-1)}{2^{2d}-1}\right)(2^{m-2}\pm 2^{\frac{m}{2}-1})\right) \\ B_3=(2^m-1)2^{m-2}\left(1+(2^m-1)\dfrac{2^{2d}-2^d-1}{2^{2d}-1}\right) \\ B_4=(2^m-1)\left(\dfrac{2^{m-2d-1}\pm 2^{\frac{m-2d}{2}-1}}{2^d+1}+\dfrac{(2^{m-d}-2^d)(2^{m-2d-2}\pm 2^{\frac{m-2d}{2}-1})}{2^{2d}-1}\right) \\ B_5=(2^m-1)\dfrac{2^{m-2d-2}(2^{m-d}-2^d)}{2^{2d}-1} \\ B_6=(2^m-1)2^{m-d}(2^{m-d-2}\pm 2^{\frac{m-d}{2}-1}) \\ B_7=(2^m-1)2^{2m-2d-2} \end{cases}$$

**Proof** Since $\dfrac{m}{\gcd(m,k)}$ is even, we get that $m$ is even. Then we have that $m-2d$ is even. And $m-d$ is odd if $d$ is odd and even if $d$ is even. Here we only give the proof for odd $d$ since the case for even $d$ can be similarly proved. The values of the exponential sum $\rho_2(a,b)=\xi(b,c,c')$ can be calculated as follows.

**Case 1** $b=c=0$. This is a trivial case, we can obtain that

$$\xi(0,0,c')=\sum_{x\in T_m}(-1)^{tr_m(c'x)}=\begin{cases}0 & c'\neq 0\\ 2^m & c'=0\end{cases}$$

**Case 2** $b=0, c\neq 0$. Note that $B_{b,c}(x,y)$ is nonalternating and $\text{rank}(B_{b,c}(x,y))=m$. Then by Lemma 1.6, when $(c,c')$ runs through $T_m^*\times T_m$, we have

$$\xi_2(0,c,c')=\begin{cases}\pm 2^{\frac{m}{2}} & (2^m-1)(2^{m-2}\pm 2^{\frac{m}{2}-1})\text{times}\\ \pm 2^{\frac{m}{2}}\sqrt{-1} & (2^m-1)2^{m-2}\text{ times(each)}\end{cases}$$

**Case 3** $b\in\langle\xi^{2^d+1}\rangle, c=0$. Note that $B_{b,c}(x,y)$ is alternating and $\text{rank}(B_{b,c}(x,y))=m-2d$. Then by Lemma 1.5, when $(c,c')$ runs through $\langle\xi^{2^d+1}\rangle\times T_m$, we have

$$\xi_2(b,0,c')=\begin{cases}0 & \dfrac{2^m-1}{2^d+1}(2^m-2^{m-2d})\text{times}\\ \pm 2^{m-\frac{m-2d}{2}} & \dfrac{2^m-1}{2^d+1}(2^{m-2d-1}\pm 2^{\frac{m-2d}{2}-1})\text{times}\end{cases}$$

**Case 4** $b\in\langle\xi^{2^d+1}\rangle, b\neq 0, c=0$. Note that

$B_{b,c}(x,y)$ is alternating and $\mathrm{rank}(B_{b,c}(x,y))=m$. Then by Lemma 1.5, when $(c,c')$ runs through $(T_m^*-\langle\xi^{2^d+1}\rangle)\times T_m$, we have

$$\xi_2(b,0,c')=\begin{cases}0 & 0 \text{ times}\\ \pm 2^{\frac{m}{2}} & \left(2^m-1-\frac{2^m-1}{2^d+1}\right)(2^{m-1}\pm 2^{\frac{m}{2}-1})\text{times}\end{cases}$$

**Case 5** $b\neq 0, c\neq 0$. For $bc\neq 0$, $\mathrm{rank}(B_{b,c}(x,y))=m, m-d$ or $m-2d$. In this case, $B_{b,c}(x,y)$ is nonalternating. From Lemma 1.5 and Theorem 3.4, we have the following results. Firstly, when $(b,c)$ runs through $R_0\setminus\{(0,c),(b,0):c\in T_m^*, b\neq 0, b\in\langle\xi^{2^d+1}\rangle\}$ and $c'$ runs through $T_m$, we have

$$\xi_2(b,c,c')=\begin{cases}\pm 2^{\frac{m}{2}} & (2^m-1)(n_0+n_2)(2^{m-2}\pm 2^{\frac{m}{2}-1})\text{times}\\ \pm 2^{\frac{m}{2}}\sqrt{-1} & (2^m-1)(n_0+n_2)2^{m-2} \text{ times(each)}\end{cases}$$

Secondly, when $(b,c)$ runs through $R_d$ and $c'$ runs through $T_m$, we have

$$\xi_2(b,c,c')=\begin{cases}0 & (2^m-1)n_1\cdot(2^m-2^{m-d})\text{times}\\ \pm(1+\sqrt{-1})2^{m-\frac{m-d+1}{2}} & (2^m-1)n_1\cdot(2^{m-d-2}\pm 2^{\frac{m-d-3}{2}})\text{times}\\ \pm(1-\sqrt{-1})2^{m-\frac{m-d+1}{2}} & (2^m-1)n_1\cdot(2^{m-d-2}\pm 2^{\frac{m-d-3}{2}})\text{times}\end{cases}$$

Thirdly, when $(b,c)$ runs through $R_{2d}\setminus\{(b,0):c\in T_m^*, b\in\langle\xi^{2^d+1}\rangle\}$ and $c'$ runs through $T_m$, we have

$$\xi_2(b,c,c')=\begin{cases}0 & (2^m-1)n_{2^d+1}\cdot(2^m-2^{m-2d})\text{times}\\ \pm 2^{m-\frac{m-2d}{2}} & (2^m-1)n_{2^d+1}\cdot(2^{m-2d-2}\pm 2^{\frac{m-2d}{2}-1})\text{times}\\ \pm 2^{m-\frac{m-2d}{2}}\sqrt{-1} & (2^m-1)n_{2^d+1}\cdot 2^{m-2d-2} \text{ times(each)}\end{cases}$$

The proof is finished by combining Cases 1—5.

### 3.2 Lee weight distribution and Hamming weight distribution of $C_2$

Now we define another class of quaternary codes as

$$C_2=\{\boldsymbol{c}(a,b):a\in R, b\in T_m\}$$

where $\boldsymbol{c}(a,b)=(Tr_m(ax+2bx^{2^k+1}))_{x\in T_m}$ and $\frac{m}{\gcd(m,k)}$ is even. Using the same techniques with that in Section 2, we have

$$w_L(\boldsymbol{c}(a,b))=2^m-\Re(\rho_2(a,b))$$

Then we can determine the Lee Hamming weight distribution of $C_2$ from Theorem 3.5 in the following.

**Theorem 3.6** When $(a,b)$ runs through $R\times T_m$, the Lee weight distributions of the quaternary code $C_2$ are given in Tables 10, 11.

**Table 10 Lee weight distribution of quaternary code $C_2$ for odd $d$**

| Weight | Frequency |
|---|---|
| 0 | 0 |
| $2^m$ | $A_1$ |
| $2^m\mp 2^{\frac{m}{2}}$ | $A_2$ |
| $2^m\mp 2^{m-\frac{m-2d}{2}}$ | $A_3$ |
| $2^m\mp 2^{m-\frac{m-d+1}{2}}$ | $A_4$ |

where

$$\begin{cases}A_1=(2^m-1)\Big(1+2^{2m-2d}(2^d-1)+2^{m-2d}(2^{m-d}-1)+2^{m-1}+\dfrac{(2^{2m-1}-2^{m-1})(2^{2d}-2^d-1)+2^{m-2d-1}(2^{m-d}-2^d)}{2^{2d}-1}\Big)\\ A_2=(2^m-1)\Big(\dfrac{2^d}{2^d+1}(2^{m-1}\pm 2^{\frac{m}{2}-1})+\Big(1+\dfrac{(2^m-1)(2^{2d}-2^d-1)}{2^{2d}-1}\Big)(2^{m-2}\pm 2^{\frac{m}{2}-1})\Big)\\ A_3=(2^m-1)\Big(\dfrac{2^{m-2d-1}\pm 2^{\frac{m-2d}{2}-1}}{2^d+1}+\dfrac{(2^{m-d}-2^d)(2^{m-2d-2}\pm 2^{\frac{m-2d}{2}-1})}{2^{2d}-1}\Big)\\ A_4=(2^m-1)2^{m-d+1}(2^{m-d-2}\pm 2^{\frac{m-d-3}{2}})\end{cases}$$

**Table 11 Lee weight distribution of quaternary code $C_2$ for even $d$**

| Weight | Frequency |
|---|---|
| 0 | 0 |
| $2^m$ | $B_1$ |
| $2^m\mp 2^{\frac{m}{2}}$ | $B_2$ |
| $2^m\mp 2^{m-\frac{m-2d}{2}}$ | $B_3$ |
| $2^m\mp 2^{m-\frac{m-d}{2}}$ | $B_4$ |

where

$$\begin{cases} B_1=(2^m-1)(1+2^{2m-2d}(2^d-1)+2^{m-2d}(2^{m-d}-1)+2^{m-1}+2^{2m-2d-1}+\dfrac{(2^{2m-1}-2^{m-1})(2^{2d}-2^d-1)+2^{m-2d-1}(2^{m-d}-2^d)}{2^{2d}-1}) \\ B_2=(2^m-1)\Big(\dfrac{2^d}{2^d+1}(2^{m-1}\pm 2^{\frac{m}{2}-1})+\Big(1+\dfrac{(2^m-1)(2^{2d}-2^d-1)}{2^{2d}-1}\Big)(2^{m-2}\pm 2^{\frac{m}{2}-1})\Big) \\ B_3=(2^m-1)\Big(\dfrac{2^{m-2d-1}\pm 2^{\frac{m-2d}{2}-1}}{2^d+1}+\dfrac{(2^{m-d}-2^d)(2^{m-2d-2}\pm 2^{\frac{m-2d}{2}-1})}{2^{2d}-1}\Big) \\ B_4=(2^m-1)2^{m-d}(2^{m-d-2}\pm 2^{\frac{m-d}{2}-1}) \end{cases}$$

By using the same method in Section 2, one can easily deduce that

$$w_{\mathrm{H}}(\boldsymbol{c}(a,b))=\begin{cases} 2^{m-1}-\dfrac{1}{2}\Re(\rho_2(a,b)) & c=0 \\ 3\cdot 2^{m-2}-\dfrac{1}{2}\Re(\rho_2(a,b)) & c\neq 0 \end{cases}$$

**Lemma 3.7** When $c=0$, the distribution of $\rho_2(a,b)$ is

$$\begin{cases} 0 & (2^m-1)(1+2^{m-2d}(2^d-1))\text{times} \\ 2^m & \text{once} \\ \pm 2^{\frac{m}{2}} & \Big(2^m-1-\dfrac{2^m-1}{2^d+1}\Big)(2^{m-1}\pm 2^{\frac{m}{2}-1})\text{times} \\ \pm 2^{m-\frac{m-2d}{2}} & \dfrac{2^m-1}{2^d+1}(2^{m-2d-1}\pm 2^{\frac{m-2d}{2}-1})\text{times} \end{cases}$$

When $c\neq 0$ and $d$ is odd, the distribution of $\rho_2(a,b)$ is

$$\begin{cases} 0 & (2^m-1)(2^{m-d}(2^m-2^{m-d})+2^{m-2d}(2^{m-d}-2^d))\text{times} \\ \pm 2^{\frac{m}{2}} & (2^m-1)(2^{m-2}\pm 2^{\frac{m}{2}-1})\Big(1+\dfrac{(2^m-1)(2^{2d}-2^d-1)}{2^{2d}-1}\Big)\text{times} \\ \pm 2^{\frac{m}{2}}\sqrt{-1} & (2^m-1)2^{m-2}\Big(1+\dfrac{(2^m-1)(2^{2d}-2^d-1)}{2^{2d}-1}\Big)(\text{times(each)} \\ \pm(1+\sqrt{-1})2^{m-\frac{m-d+1}{2}} & (2^m-1)2^{m-d}(2^{m-d-2}\pm 2^{\frac{m-d-3}{2}})\text{times} \\ \pm(1-\sqrt{-1})2^{m-\frac{m-d+1}{2}} & (2^m-1)2^{m-d}(2^{m-d-2}\pm 2^{\frac{m-d-3}{2}})\text{times} \\ \pm 2^{m-\frac{m-2d}{2}} & (2^m-1)\dfrac{2^{m-d}-2^d}{2^{2d}-1}(2^{m-2d-2}\pm 2^{\frac{m-2d}{2}-1})\text{times} \\ \pm 2^{m-\frac{m-2d}{2}}\sqrt{-1} & (2^m-1)\dfrac{2^{m-d}-2^d}{2^{2d}-1}2^{m-2d-2}\ \text{times} \end{cases}$$

When $c\neq 0$ and $d$ is even, the distribution of $\rho_2(a,b)$ is

$$\begin{cases} 0 & (2^m-1)(2^{m-d}(2^m-2^{m-d})+2^{m-2d}(2^{m-d}-2^d))\text{times} \\ \pm 2^{\frac{m}{2}} & (2^m-1)(2^{m-2}\pm 2^{\frac{m}{2}-1})(1+\dfrac{(2^m-1)(2^{2d}-2^d-1)}{2^{2d}-1})\text{times} \\ \pm 2^{\frac{m}{2}}\sqrt{-1} & (2^m-1)2^{m-2}(1+\dfrac{(2^m-1)(2^{2d}-2^d-1)}{2^{2d}-1})\text{times(each)} \\ \pm 2^{m-\frac{m-d}{2}} & (2^m-1)2^{m-d}(2^{m-d-2}\pm 2^{\frac{m-d}{2}-1})\text{times} \\ \pm 2^{m-\frac{m-d}{2}}\sqrt{-1} & (2^m-1)2^{2m-2d-2}\text{times(each)} \\ \pm 2^{m-\frac{m-2d}{2}} & (2^m-1)\dfrac{2^{m-d}-2^d}{2^{2d}-1}(2^{m-2d-2}\pm 2^{\frac{m-2d}{2}-1})\text{times} \\ \pm 2^{m-\frac{m-2d}{2}}\sqrt{-1} & (2^m-1)\dfrac{2^{m-d}-2^d}{2^{2d}-1}2^{m-2d-2}\text{times(each)} \end{cases}$$

**Proof** The proof is completed by the proof of Theorem 3.5.

**Theorem 3.8** When $(a,b)$ runs through $R\times T_m$, the Hamming weight distributions of the quaternary code $C_2$ are given in Tables 12, 13.

**Table 12 Hamming weight distribution of quaternary code $C_2$ for odd $d$**

| Weight | Frequency |
|---|---|
| 0 | 1 |
| $2^{m-1}$ | $(2^m-1)\cdot(1+2^{m-2d}(2^d-1))$ |
| $2^{m-1}\mp 2^{\frac{m}{2}-1}$ | $(2^m-1)\cdot\frac{2^d(2^{m-1}\pm 2^{\frac{m}{2}-1})}{2^d+1}$ |
| $2^{m-1}\mp 2^{m-\frac{m-2d}{2}-1}$ | $\frac{2^m-1}{2^d+1}\cdot(2^{m-2d-1}\pm 2^{\frac{m-2d}{2}-1})$ |
| $3\cdot 2^{m-2}$ | $A_1$ |
| $3\cdot 2^{m-2}\mp 2^{\frac{m}{2}-1}$ | $A_2$ |
| $3\cdot 2^{m-2}\mp 2^{m-\frac{m-d+1}{2}-1}$ | $A_3$ |
| $3\cdot 2^{m-2}\mp 2^{m-\frac{m-2d}{2}-1}$ | $A_4$ |

**Table 13 Hamming weight distribution of quaternary code $C_2$ for even $d$**

| Weight | Frequency |
|---|---|
| 0 | 1 |
| $2^{m-1}$ | $(2^m-1)\cdot(1+2^{m-2d}(2^d-1))$ |
| $2^{m-1}\mp 2^{\frac{m}{2}-1}$ | $(2^m-1)\cdot\frac{2^d(2^{m-1}\pm 2^{\frac{m}{2}-1})}{2^d+1}$ |
| $2^{m-1}\mp 2^{m-\frac{m-2d}{2}-1}$ | $\frac{2^m-1}{2^d+1}\cdot(2^{m-2d-1}\pm 2^{\frac{m-2d}{2}-1})$ |
| $3\cdot 2^{m-2}$ | $B_1$ |
| $3\cdot 2^{m-2}\mp 2^{\frac{m}{2}-1}$ | $B_2$ |
| $3\cdot 2^{m-2}\mp 2^{m-\frac{m-d}{2}-1}$ | $B_3$ |
| $3\cdot 2^{m-2}\mp 2^{m-\frac{m-2d}{2}-1}$ | $B_4$ |

where

$$\begin{cases} A_1=(2^m-1)(2^{m-d}(2^m-2^{m-d})+2^{m-2d}(2^{m-d}-2^d)+2^{m-1}+ \\ \quad \frac{(2^{2m-1}-2^{m-1})(2^{2d}-2^d-1)+2^{m-2d-1}(2^{m-d}-2^d)}{2^{2d}-1}) \\ A_2=(2^m-1)(2^{m-2}\pm 2^{\frac{m}{2}-1})\left(1+\frac{(2^m-1)(2^{2d}-2^d-1)}{2^{2d}-1}\right) \\ A_3=(2^m-1)2^{m-d+1}(2^{m-d-2}\pm 2^{\frac{m-d-3}{2}}) \\ A_4=(2^m-1)\,\frac{2^{m-d}-2^d}{2^{2d}-1}(2^{m-2d-2}\pm 2^{\frac{m-2d}{2}-1}) \end{cases}$$

$$\begin{cases} B_1=(2^m-1)(2^{m-d}(2^m-2^{m-d})+2^{m-2d}(2^{m-d}-2^d)+2^{m-1}+ \\ \quad 2^{2m-2d-1}+\frac{(2^{2m-1}-2^{m-1})(2^{2d}-2^d-1)+2^{m-2d-1}(2^{m-d}-2^d)}{2^{2d}-1}) \\ B_2=(2^m-1)(2^{m-2}\pm 2^{\frac{m}{2}-1})\left(1+\frac{(2^m-1)(2^{2d}-2^d-1)}{2^{2d}-1}\right) \\ B_3=(2^m-1)2^{m-d}(2^{m-d-2}\pm 2^{\frac{m-d}{2}-1}) \\ B_4=(2^m-1)\,\frac{2^{m-d}-2^d}{2^{2d}-1}(2^{m-2d-2}\pm 2^{\frac{m-2d}{2}-1}) \end{cases}$$

**Proof** The proof is completed by Lemma 3.7.

### 3.3 Complete weight distribution of $C_2$

Let $N_i, i=0,1,2,3$, denote the number of components of $\boldsymbol{c}(a,b)$ in $C_2$ that are equal to $i$. Note that

$$N_0=2^m-w_{\mathrm{H}}(\boldsymbol{c}(a,b))=$$

$$\begin{cases} 2^{m-1}+\frac{1}{2}\Re(\rho_2(a,b)) & c=0 \\ 2^{m-2}+\frac{1}{2}\Re(\rho_2(a,b)) & c\neq 0 \end{cases}$$

$$N_1=\begin{cases} 0 & c=0 \\ 2^{m-2}+\frac{1}{2}\Im(\rho_2(a,b)) & c\neq 0 \end{cases}$$

$$N_2=w_{\mathrm{L}}(\boldsymbol{c}(a,b))-w_{\mathrm{H}}(\boldsymbol{c}(a,b))=$$

$$\begin{cases} 2^{m-1}-\frac{1}{2}\Re(\rho_2(a,b)) & c=0 \\ 2^{m-2}-\frac{1}{2}\Re(\rho_2(a,b)) & c\neq 0 \end{cases}$$

$$N_4=\begin{cases} 0 & c=0 \\ 2^{m-2}-\frac{1}{2}\Im(\rho_2(a,b)) & c\neq 0 \end{cases}$$

In the following, we give the distributions of $(N_0, N_1, N_2, N_3)$ when $(a,b)$ runs through $R\times T_m$.

**Theorem 3.9** The complete weight enumerator of the quaternary code $C_2$ is given by Table 14 if $d$ is odd and Table 15 if $d$ is even when $(a,b)$ runs through $R\times T_m$.

**Table 14 Complete weight enumerator of $C_2$ for odd $d$**

| $N_0$ | $N_1$ | $N_2$ | $N_3$ | Frequency |
|---|---|---|---|---|
| $2^{m-1}$ | $2^{m-1}$ | 0 | 0 | $A_1$ |
| $2^m$ | 0 | 0 | 0 | $A_2$ |
| $2^{m-1}\pm 2^{\frac{m}{2}-1}$ | $2^{m-1}\mp 2^{\frac{m}{2}-1}$ | 0 | 0 | $A_3$ |
| $2^{m-1}\pm 2^{m-\frac{m-d+1}{2}-1}$ | $2^{m-1}\mp 2^{m-\frac{m-d+1}{2}-1}$ | 0 | 0 | $A_4$ |
| $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}$ | $A_5$ |
| $2^{m-2}\pm 2^{\frac{m}{2}-1}$ | $2^{m-2}\mp 2^{\frac{m}{2}-1}$ | $2^{m-2}$ | $2^{m-2}$ | $A_6$ |
| $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}\pm 2^{\frac{m}{2}-1}$ | $2^{m-2}\mp 2^{\frac{m}{2}-1}$ | $A_7$ |
| $2^{m-2}\pm 2^{m-\frac{m-d+1}{2}-1}$ | $2^{m-2}\mp 2^{m-\frac{m-d+1}{2}-1}$ | $2^{m-2}\pm 2^{m-\frac{m-d+1}{2}-1}$ | $2^{m-2}\mp 2^{m-\frac{m-d+1}{2}-1}$ | $A_8$ |
| $2^{m-2}\pm 2^{m-\frac{m-d+1}{2}-1}$ | $2^{m-2}\mp 2^{m-\frac{m-d+1}{2}-1}$ | $2^{m-2}\mp 2^{m-\frac{m-d+1}{2}-1}$ | $2^{m-2}\pm 2^{m-\frac{m-d+1}{2}-1}$ | $A_9$ |
| $2^{m-2}\pm 2^{m-\frac{m-2d+1}{2}-1}$ | $2^{m-2}\mp 2^{m-\frac{m-2d+1}{2}-1}$ | $2^{m-2}$ | $2^{m-2}$ | $A_{10}$ |
| $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}\pm 2^{m-\frac{m-2d+1}{2}-1}$ | $2^{m-2}\mp 2^{m-\frac{m-2d+1}{2}-1}$ | $A_{11}$ |

where

$$A_1=(2^m-1)(1+2^{m-2d}(2^d-1))$$

$$A_2=1$$

$$A_3=\left(2^m-1-\frac{2^m-1}{2^d+1}\right)(2^{m-1}\pm 2^{\frac{m}{2}-1})$$

$$A_4=\frac{2^m-1}{2^d+1}(2^{m-2d-1}\pm 2^{\frac{m-2d}{2}-1})$$

$$A_5=(2^m-1)(2^{m-d}(2^m-2^{m-d})+2^{m-2d}(2^{m-d}-2^d))$$

$$A_6=(2^m-1)(2^{m-2}\pm 2^{\frac{m}{2}-1})\cdot\left(1+\frac{(2^m-1)(2^{2d}-2^d-1)}{2^{2d}-1}\right)$$

$$A_7=(2^m-1)2^{m-2}\left(1+\frac{(2^m-1)(2^{2d}-2^d-1)}{2^{2d}-1}\right)\text{(each)}$$

$$A_8=(2^m-1)2^{m-d}(2^{m-d-2}\pm 2^{\frac{m-d-3}{2}})$$

$$A_9=(2^m-1)2^{m-d}(2^{m-d-2}\pm 2^{\frac{m-d-3}{2}})$$

$$A_{10}=(2^m-1)\frac{2^{m-d}-2^d}{2^{2d}-1}(2^{m-2d-2}\pm 2^{\frac{m-2d}{2}-1})$$

$$A_{11}=(2^m-1)\frac{2^{m-d}-2^d}{2^{2d}-1}2^{m-2d-2}\text{(each)}$$

**Table 15 Complete weight enumerator of $C_2$ for even $d$**

| $N_0$ | $N_1$ | $N_2$ | $N_3$ | Frequency |
|---|---|---|---|---|
| $2^{m-1}$ | $2^{m-1}$ | 0 | 0 | $B_1$ |
| $2^m$ | 0 | 0 | 0 | $B_2$ |
| $2^{m-1}\pm 2^{\frac{m}{2}-1}$ | $2^{m-1}\mp 2^{\frac{m}{2}-1}$ | 0 | 0 | $B_3$ |
| $2^{m-1}\pm 2^{m-\frac{m-2d}{2}-1}$ | $2^{m-1}\mp 2^{m-\frac{m-2d}{2}-1}$ | 0 | 0 | $B_4$ |
| $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}$ | $B_5$ |
| $2^{m-2}\pm 2^{\frac{m}{2}-1}$ | $2^{m-2}\mp 2^{\frac{m}{2}-1}$ | $2^{m-2}$ | $2^{m-2}$ | $B_6$ |
| $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}\pm 2^{\frac{m}{2}-1}$ | $2^{m-2}\mp 2^{\frac{m}{2}-1}$ | $B_7$ |
| $2^{m-2}\pm 2^{m-\frac{m-d}{2}-1}$ | $2^{m-2}\mp 2^{m-\frac{m-d}{2}-1}$ | $2^{m-2}$ | $2^{m-2}$ | $B_8$ |
| $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}\pm 2^{m-\frac{m-d}{2}-1}$ | $2^{m-2}\mp 2^{m-\frac{m-d}{2}-1}$ | $B_9$ |
| $2^{m-2}\pm 2^{m-\frac{m-2d}{2}-1}$ | $2^{m-2}\mp 2^{m-\frac{m-2d}{2}-1}$ | $2^{m-2}$ | $2^{m-2}$ | $B_{10}$ |
| $2^{m-2}$ | $2^{m-2}$ | $2^{m-2}\pm 2^{m-\frac{m-2d}{2}-1}$ | $2^{m-2}\mp 2^{m-\frac{m-2d}{2}-1}$ | $B_{11}$ |

where

$$\begin{cases} B_1 = 2^m - 1 \\ B_2 = 1 \\ B_3 = \dfrac{(2^m-1)(2^{2d}-2^d-1)(2^{m-2} \pm 2^{\frac{m}{2}-1})}{2^{2d}-1} \\ B_4 = \dfrac{(2^m-1)(2^{2d}-2^d-1)2^{m-2}}{2^{2d}-1} \text{(each)} \\ B_5 = 2^{m-d}(2^m - 2^{m-d}) + 2^{m-2d}(2^{m-d} - 2^d) \\ B_6 = 2^{m-d}(2^{m-d-2} \pm 2^{\frac{m-d-3}{2}}) \\ B_7 = 2^{m-d}(2^{m-d-2} \pm 2^{\frac{m-d-3}{2}}) \\ B_8 = (2^m-1)2^{m-d}(2^{m-d-2} \pm 2^{\frac{m-d}{2}-1}) \\ B_9 = (2^m-1)2^{2m-2d-2} \\ B_{10} = (2^m-1)\dfrac{2^{m-d}-2^d}{2^{2d}-1}(2^{m-2d-2} \pm 2^{\frac{m-2d}{2}-1}) \\ B_{11} = (2^m-1)\dfrac{2^{m-d}-2^d}{2^{2d}-1}2^{m-2d-2} \text{(each)} \end{cases}$$

**Proof** The proof is completed by Lemma 3.7.

## 4 Conclusions

Two classes of quaternary codes $C_1$ and $C_2$ based on two exponential sums $\rho_1(a,b)$ and $\rho_2(a,b)$ over Galois rings are investigated, respectively. The Lee weight distributions, Hamming weight distributions and complete weight distributions of the codes are completely determined. All the distributions in this paper have been verified by computer experiments.

**References:**


[1] NECHAEV A A. The Kerdock code in a cyclic form [J]. Discrete Mathematics, 1989, 1: 123-139.

[2] HAMMONS A R, KUMAR P V, CALDERBANK A R, et al. The $\mathbb{Z}_4$-linearity of Kerdoek, Preparata, Goethals, and related codes[J]. IEEE Transaction on Information Theory, 1994,40(5): 301-319.

[3] LI N, TANG X, HELLESETH T. Several classes of codes and sequences derived from a $\mathbb{Z}_4$-valued quadratic form[J]. IEEE Transaction on Information Theory, 2011, 57(11): 7618-7628.

[4] FANY, LING S, LIU H. Matrix product codes over finite commutative Frobenius rings [J]. Designs Codes and Cryptography, 2014, 71(2): 201-227.

[5] FANY, GAO Y. Codes over algebraic integer rings of cyclotomic fields[J]. IEEE Transaction on Information Theory, 2004,50(1): 194-200.

[6] KUMAR P V, HELLESETH T, CALDERBANK A, et al. Large families of quaternary sequences with low correlation[J]. IEEE Transaction on Information Theory, 1996, 42(2): 579-592.

[7] TANG H X, HELLESETH T, FAN P. Quadriphase sequences obtained from binary quadratic form sequences[J]. Lecture Notes in Computer Science, 2005, 3486: 243-254.

[8] ZENG X, LI N, HU L. A class of nonbinary codes and sequence families[J]. Lecture Notes in Computer Science, 2008, 5203: 81-94.

[9] ZHU S, KAI X. A class of constacyclic codes over $\mathbb{Z}_{p^m}$[J]. Finite Fields and their Applications, 2010, 16(4): 243-254.

[10] ZHU S, WANG Y, SHI M. Some results on cyclic codes over $\mathbb{F}_2+v\mathbb{F}_2$[J]. IEEE Transaction on Information Theory,2010, 56(4): 1680-1684.

[11] BROWN E H. Generalizations of the Kervaire invariant[J]. Annals of Mathematics, 1972, 95(2): 368-383.

[12] SCHIMIDT K-U. $\mathbb{Z}_4$-valued quadratic forms and exponential sums[C]// IEEE International Symposium on Information Theory 2008. Toronto, Canada: [s. n.], 2008:6-11.

[13] SCHMIDT K-U. $\mathbb{Z}_4$-valued quadratic forms and quaternary sequence families[J]. IEEE Transaction on Information Theory, 2009, 55(12): 5803-5810.

[14] HELLESETH T, KUMAR P V. Codes with the same weight distribution as the Goethals codes and the Delsatrte-Goethals codes[J]. Designs Codes and Cryptography, 1996, 9(3): 257-266.

[15] SCHMIDT K U. On the correlation distribution of Delsarte-Goethals sequences[J]. Designs Codes and Cryptography, 2011, 59(1): 333-347.

[16] WOOD J A. Witt's extension theorem for mod four valued quadratic forms [J]. Transactions of the American Mathematical Society, 1993, 336(1): 445-461.

[17] HELLESETHT, KUMAR P V. Sequences with low correlation[M]. Netherlands: Elsevier, 1998.

[18] BLUHER A. On $x^{q+1}+ax+b$[J]. Finite Fields and their Applications, 2004, 10(3): 285-305.



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