Discrete Fractional Lagrange Equations of Nonconservative Systems

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Abstract: In order to study discrete nonconservative system, Hamilton's principle within fractional difference operators of Riemann-Liouville type is given. Discrete Lagrange equations of the nonconservative system as well as the nonconservative system with dynamic constraint are established within fractional difference operators of Riemann-Liouville type from the view of time scales. Firstly, time scale calculus and fractional calculus are reviewed. Secondly, with the help of the properties of time scale calculus, discrete Lagrange equation of the nonconservative system within fractional difference operators of Riemann-Liouville type is presented. Thirdly, using the Lagrange multipliers, discrete Lagrange equation of the nonconservative system with dynamic constraint is also established. Then two special cases are discussed. Finally, two examples are devoted to illustrate the results.

Key words: discrete Lagrange equation; time scale; fractional difference operator; nonconservative system

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0 Introduction

In 1937, Fort[1] first introduced the theory for the discrete calculus of variations. Based on this theory, fractional difference operators within Caputo sense were established and used to solve some difference equations[2-3]. Besides, some important results of discrete calculus of variations were summarized in Ref.[4]. Considering the useful applications of the discrete analogues of differential equations[4-5], and intense investigations on the continuous fractional calculus of variations[6-22], Bastos[23] started a fractional discrete-time theory of the calculus of variations in 2012. He introduced the fractional difference operators of Riemann-Liouville type on the basis of Refs.[24-25], and achieved the fractional discrete Euler-Lagrange equations. In particular, when $\alpha=1$, the classical discrete results of the calculus of variations can be obtained.

In this paper, we establish discrete Lagrange equations of the nonconservative system and the nonconservative system with dynamic constraint within fractional difference operators of Riemann-Liouville type. We use some properties of time scale calculus for convenience. Time scale T, which is an arbitrary nonempty closed subset of the real numbers, was introduced by Hilger in 1988[26]. It follows from the definition that time scale calculus has the features of unification and extension. From some properties of time scale T, we can obtain the corresponding properties for the continuous analysis when letting $T=\boldsymbol{R}$. Similarly, we can obtain the corresponding properties for the discrete analysis when letting $\mathrm{T}=Z$. Apart from R and Z, T has many other values, for instance, $T=q^{N_0}(q>1)$. We mainly use the properties of time scale calculus by letting $T=Z$ in this paper.

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1 Time Scale Calculus and Fractional Calculus

We briefly review time scale calculus and fractional calculus. Refs. [23, 27-28] provide more details.

A time scale T is an arbitrary nonempty closed subset of the real number set. Hence, the integer set Z and the real number set R are the special cases of T.

Let T be a time scale, then

(1) The mapping $\sigma: T \rightarrow T$, $\sigma(t)=\inf\{s \in T: s > t\}$ is called the forward jump operator.

(2) The mapping $\rho: T \rightarrow T$, $\rho(t)=\sup\{s \in T: s < t\}$ is called the backward jump operator.

(3) the mapping $\theta: T \rightarrow [0, \infty)$, $\theta(t)=\sigma(t)-t$ is called the forward graininess function.

(4) $T^{\kappa}=T \backslash (\rho(\sup T), \sup T]$ when $\sup T < \infty$; $T^{\kappa}=T$ when $\sup T=\infty$.

(5) Let $f: \boldsymbol{T} \rightarrow \boldsymbol{R}$, $t \in \boldsymbol{T}^{\kappa}$, if for any $\varepsilon > 0$, there exists $N=(t-\delta, t+\delta) \cap T$ for some $\delta > 0$ such that $\left| (f(\sigma(t))-f(\omega))-f^{\Delta}(t)(\sigma(t)-\omega) \right| \leqslant \varepsilon \left| \sigma(t)-\omega \right|$ for all $\omega \in N$, then $f^{\Delta}(t)$ is called the delta derivative of f at t.

When $T=R$, $\sigma(t)=\rho(t)=t$, $\theta(t)=0$, $f^{\Delta}(t)=\dot{f}(t)$. When $T=Z$, $\sigma(t)=t+1$, $\rho(t)=t-1$, $\theta(t)=1$, $f^{\Delta}(t)=f(t+1)-f(t)=f^{\sigma}-f=\Delta_d f$.

In this paper, a is set to be an arbitrary real number, and the time scale is $\{a, a+1, \cdots, b\}$. Then it is easy to obtain $T^{\kappa}=\{a, a+1, \cdots, b-1\}$. Let α, β be two arbitrary real numbers such that $\alpha, \beta \in (0, 1]$, and put $\mu=1-\alpha$, $\nu=1-\beta$.

For arbitrary $x, y \in R$, $x^{(y)}=\dfrac{\Gamma(x+1)}{\Gamma(x+1-y)}$, where Γ is the gamma function.

The left fractional sum and the right fractional sum are defined as

$$ {}_a\Delta_t^{\alpha} f(t)=\Delta_d({}_a\Delta_t^{-\mu} f(t))={}_a\Delta_t^{-\mu}\Delta_d f(t)+\frac{(t+\mu-a)^{(\mu-1)}}{\Gamma(\mu)} f(a) \tag{1} $$

$$ {}_t\Delta_b^{\beta} f(t)=-\Delta_d({}_t\Delta_b^{-\nu} f(t))=-{}_t\Delta_{\rho(b)}^{-\nu}\Delta_d f(t)+\frac{\nu}{\Gamma(\nu+1)}(b+\nu-\sigma(t))^{(\nu-1)} f(b) \tag{2} $$

where

$$ {}_a\Delta_t^{-\mu} f(t)=f(t)+\frac{\mu}{\Gamma(\mu+1)}\sum_{s=a}^{t-1}(t+\mu-\sigma(s))^{(\mu-1)} f(s) \quad t \in T \tag{3} $$

$$ {}_t\Delta_b^{-\nu} f(t)=f(t)+\frac{\nu}{\Gamma(\nu+1)}\sum_{s=\sigma(t)}^{b}\left(s+\nu-\sigma(t)\right)^{(\nu-1)} f(s) \quad t \in T \tag{4} $$

Hence

$$ {}_a\Delta_t^0 f(t)={}_t\Delta_b^0 f(t)=f(t), $$
$$ {}_a\Delta_t^1 f(t)=\Delta_d f(t), {}_t\Delta_b^1 f(t)=-\Delta_d f(t) \tag{5} $$

Fractional summation by parts is given as

$$ \sum_{t=a}^{b-1} f(t)\,{}_a\Delta_t^{\alpha} g(t)=f(b-1)g(b)-f(a)g(a)+\sum_{t=a}^{b-2}{}_t\Delta_{\rho(b)}^{\alpha} f(t)g^{\sigma}(t)+\frac{\mu}{\Gamma(\mu+1)}g(a)\left(\sum_{t=a}^{b-1}(t+\mu-a)^{(\mu-1)}f(t)-\sum_{t=\sigma(a)}^{b-1}(t+\mu-\sigma(a))^{(\mu-1)}f(t)\right) \tag{6} $$

$$ \sum_{t=a}^{b-1} f(t)\,{}_t\Delta_b^{\beta} g(t)=-g(b)\,{}_a\Delta_t^{-\nu} f(t)\Big|_{t=\rho(b)}+g(a)\,{}_a\Delta_t^{-\nu} f(t)\Big|_{t=a}+\sum_{t=a}^{b-2} g^{\sigma}(t)\,{}_a\Delta_t^{\beta} f(t)+\frac{\nu g(b)}{\Gamma(\nu+1)}\sum_{t=a}^{b-1}(b+\nu-\sigma(t))^{(\nu-1)} f(t) \tag{7} $$

The commutative relations between the isochronous variation and the fractional difference operators are

$$ \begin{cases} \delta\, {}_a\Delta_t^{\alpha} f(t)={}_a\Delta_t^{\alpha}\delta f(t) \\ \delta\, {}_t\Delta_b^{\beta} f(t)={}_t\Delta_b^{\beta}\delta f(t) \end{cases} \tag{8} $$

2 Discrete Equation

Assume that the configuration of a mechanical system is determined by the generalized coordinates q_i^{σ}, $i=1,2,\cdots,n$, the kinetic energy function is $\widetilde{T}=\widetilde{T}(t, q_i^{\sigma}, {}_a\Delta_t^{\alpha} q_i, {}_t\Delta_b^{\beta} q_i)$. The Hamilton's principle for the nonconservative system with fractional difference operators of Riemann-Liouville type has the following form

$$ \sum_{t=a}^{b-1}(\delta\widetilde{T}+Q_j\delta q_j^{\sigma})=0 \quad j=1,2,\cdots,n \tag{9} $$

where $Q_j\delta q_j^{\sigma}$ is the virtual work of the generalized force Q_j, $q_j(a)=A_j$, $q_j(b)=B_j$.

From Eqs. (6) and (7), we have

$$\sum_{t=a}^{b-1}\frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j}\cdot\delta\, {}_a\Delta_t^\alpha q_j=\sum_{t=a}^{b-1}\frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j}\cdot {}_a\Delta_t^\alpha \delta q_j=$$
$$\frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j}(b-1)\cdot\delta q_j(b)-\frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j}(a)\cdot\delta q_j(a)+$$
$$\sum_{t=a}^{b-2}{}_t\Delta_{\rho(b)}^\alpha\frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j}(t)\cdot\delta q_j^\sigma(t)+\frac{\mu\delta q_j(a)}{\Gamma(\mu+1)}\times$$
$$\left[\sum_{t=a}^{b-1}(t+\mu-a)^{(\mu-1)}\frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j}(t)-\right.$$
$$\left.\sum_{t=\sigma(a)}^{b-1}(t+\mu-\sigma(a))^{(\mu-1)}\frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j}(t)\right] \quad (10)$$

$$\sum_{t=a}^{b-1}\frac{\partial \widetilde{T}}{\partial\, {}_t\Delta_b^\beta q_j}\cdot\delta\, {}_t\Delta_b^\beta q_j=\sum_{t=a}^{b-1}\frac{\partial \widetilde{T}}{\partial\, {}_t\Delta_b^\beta q_j}\cdot {}_t\Delta_b^\beta \delta q_j=$$
$$-\delta q_j(b)\cdot {}_a\Delta_t^{-\nu}\frac{\partial \widetilde{T}}{\partial\, {}_t\Delta_b^\beta q_j}(t)\Big|_{t=\rho(b)}+$$
$$\delta q_j(a)\cdot {}_a\Delta_t^{-\nu}\frac{\partial \widetilde{T}}{\partial\, {}_t\Delta_b^\beta q_j}(t)\Big|_{t=a}+$$
$$\sum_{t=a}^{b-2}\delta q_j^\sigma(t)\cdot {}_a\Delta_t^\beta\frac{\partial \widetilde{T}}{\partial\, {}_t\Delta_b^\beta q_j}(t)+\frac{\nu\delta q_j(b)}{\Gamma(\nu+1)}\cdot$$
$$\sum_{t=a}^{b-1}(b+\nu-\sigma(t))^{(\nu-1)}\frac{\partial \widetilde{T}}{\partial\, {}_t\Delta_b^\beta q_j}(t) \quad (11)$$

Considering

$$\sum_{t=a}^{b-1}\frac{\partial \widetilde{T}}{\partial q_j^\sigma}\cdot\delta q_j^\sigma=\frac{\partial \widetilde{T}}{\partial q_j^\sigma}\cdot\delta q_j^\sigma\Big|_{t=b-1}+\sum_{t=a}^{b-2}\frac{\partial \widetilde{T}}{\partial q_j^\sigma}\cdot\delta q_j^\sigma \quad (12)$$

$$\sum_{t=a}^{b-1}Q_j\cdot\delta q_j^\sigma=Q_j\cdot\delta q_j^\sigma\Big|_{t=b-1}+\sum_{t=a}^{b-2}Q_j\cdot\delta q_j^\sigma \quad (13)$$

and the boundary conditions $q_j(a)=A_j$, $q_j(b)=B_j$, we have

$$\sum_{t=a}^{b-1}(\delta\widetilde{T}+Q_j\delta q_j^\sigma)=\sum_{t=a}^{b-1}\left(\frac{\partial \widetilde{T}}{\partial q_j^\sigma}\cdot\delta q_j^\sigma+\right.$$
$$\left.\frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j}\delta\, {}_a\Delta_t^\alpha q_j+\frac{\partial \widetilde{T}}{\partial\, {}_t\Delta_b^\beta q_j}\cdot\delta\, {}_t\Delta_b^\beta q_j+Q_j\delta q_j^\sigma\right)=$$
$$\sum_{t=a}^{b-2}\left(\frac{\partial \widetilde{T}}{\partial q_j^\sigma}+{}_t\Delta_{\rho(b)}^\alpha\frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j}+{}_a\Delta_t^\beta\frac{\partial \widetilde{T}}{\partial\, {}_t\Delta_b^\beta q_j}+Q_j\right)\delta q_j^\sigma=0 \quad (14)$$

Since the value of δq_j^σ is arbitrary, we obtain

$$\frac{\partial \widetilde{T}}{\partial q_j^\sigma}+{}_t\Delta_{\rho(b)}^\alpha\frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j}+{}_a\Delta_t^\beta\frac{\partial \widetilde{T}}{\partial\, {}_t\Delta_b^\beta q_j}+Q_j=0$$
$$t\in\{a,a+1,\cdots,b-2\} \quad (15)$$

In Eq.(15), Q_j contains the conservative force Q'_j and the nonconservative force Q''_j. If Q'_j is potential, that is, there exists a function $V=V(t,q_i^\sigma)$ such that

$$Q'_j=-\frac{\partial V}{\partial q_j^\sigma} \quad (16)$$

Since

$$\frac{\partial V}{\partial\, {}_a\Delta_t^\alpha q_j}=0,\ \frac{\partial V}{\partial\, {}_t\Delta_b^\beta q_j}=0 \quad (17)$$

Substituting Eqs.(16) and (17) into Eq.(15), we have

$$\frac{\partial(\widetilde{T}-V)}{\partial q_j^\sigma}+{}_t\Delta_{\rho(b)}^\alpha\frac{\partial(\widetilde{T}-V)}{\partial\, {}_a\Delta_t^\alpha q_j}+{}_a\Delta_t^\beta\frac{\partial(\widetilde{T}-V)}{\partial\, {}_t\Delta_b^\beta q_j}+$$
$$Q''_j=0\quad t\in\{a,a+1,\cdots,b-2\} \quad (18)$$

Let $L=\tilde{T}-V$, Eq.(18) can be written as

$$\frac{\partial L}{\partial q_j^\sigma}+{}_t\Delta_{\rho(b)}^\alpha\frac{\partial L}{\partial\, {}_a\Delta_t^\alpha q_j}+{}_a\Delta_t^\beta\frac{\partial L}{\partial\, {}_t\Delta_b^\beta q_j}+Q''_j=0$$
$$t\in\{a,a+1,\cdots,b-2\} \quad (19)$$

If Q'_j has the generalized potential, that is, there exists a function $U=U(t,q_i^\sigma,{}_a\Delta_t^\alpha q_i,{}_t\Delta_b^\beta q_i)$ such that

$$Q'_j=\frac{\partial U}{\partial q_j^\sigma}+{}_t\Delta_{\rho(b)}^\alpha\frac{\partial U}{\partial\, {}_a\Delta_t^\alpha q_j}+{}_a\Delta_t^\beta\frac{\partial U}{\partial\, {}_t\Delta_b^\beta q_j} \quad (20)$$

Substituting Eq.(20) into Eq.(15), we have

$$\frac{\partial(\widetilde{T}+U)}{\partial q_j^\sigma}+{}_t\Delta_{\rho(b)}^\alpha\frac{\partial(\widetilde{T}+U)}{\partial\, {}_a\Delta_t^\alpha q_j}+{}_a\Delta_t^\beta\frac{\partial(\widetilde{T}+U)}{\partial\, {}_t\Delta_b^\beta q_j}+$$
$$Q''_j=0\quad t\in\{a,a+1,\cdots,b-2\} \quad (21)$$

Let $L=\widetilde{T}+U=\widetilde{T}-V$, Eq. (21) can be written as

$$\frac{\partial L}{\partial q_j^\sigma}+{}_t\Delta_{\rho(b)}^\alpha\frac{\partial L}{\partial\, {}_a\Delta_t^\alpha q_j}+{}_a\Delta_t^\beta\frac{\partial L}{\partial\, {}_t\Delta_b^\beta q_j}+Q''_j=0$$
$$t\in\{a,a+1,\cdots,b-2\} \quad (22)$$

Eq.(22) is called discrete fractional Lagrange equation of the nonconservative system.

Remark 1 If $\alpha=1$, L does not depend on ${}_t\Delta_b^\beta q_i$, and the discrete Lagrange equation of the nonconservative system can be obtained

$$\frac{\partial L(t,q_i(t+1),\Delta_d q_i)}{\partial q_j(t+1)}-\Delta_d\frac{\partial L}{\partial\Delta_d q_j}+Q''_j=0$$
$$t\in\{a,a+1,\cdots,b-2\} \quad (23)$$

Eq.(23) is consistent with the result in Ref.[28].

3 Discrete Equation with Dynamic Constraint

We assume that the motion of the nonconservative system is subjected to the following ideal dynamic constraint

$$h_k(t, q_i^\sigma, {}_a\Delta_t^\alpha q_i, {}_t\Delta_b^\beta q_i) = 0$$
$$k = 1, 2, \cdots, g;\ i = 1, 2, \cdots, n \tag{24}$$

which satisfies

$$\frac{\partial h_k}{\partial\, {}_a\Delta_t^\alpha q_j}\delta q_j^\sigma = 0, \frac{\partial h_k}{\partial\, {}_t\Delta_b^\beta q_j}\delta q_j^\sigma = 0 \tag{25}$$

In the sequel, we study the d'Alembert-Lagrange principle with fractional difference operators. By virtue of Eq.(15), the universal d'Alembert-Lagrange principle with fractional difference operators can be expressed as

$$\left(\frac{\partial \widetilde{T}}{\partial q_j^\sigma} + {}_t\Delta_{\rho(b)}^\alpha \frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j} + {}_a\Delta_t^\beta \frac{\partial \widetilde{T}}{\partial\, {}_t\Delta_b^\beta q_j} + Q_j\right)\cdot$$
$$[\delta q\sigma j = 0][t \in a, a+1, \cdots, b-2] \tag{26}$$

Introducing the Lagrange multipliers λ_k, $k = 1, 2, \cdots, g$, from Eq.(25), we obtain

$$\lambda_k \frac{\partial h_k}{\partial\, {}_a\Delta_t^\alpha q_j}\delta q_j^\sigma = 0, \lambda_k \frac{\partial h_k}{\partial\, {}_t\Delta_b^\beta q_j}\delta q_j^\sigma = 0 \tag{27}$$

It follows from Eqs.(26) and (27) that

$$\left(\frac{\partial \widetilde{T}}{\partial q_j^\sigma} + {}_t\Delta_{\rho(b)}^\alpha \frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j} + {}_a\Delta_t^\beta \frac{\partial \widetilde{T}}{\partial\, {}_t\Delta_b^\beta q_j} + Q_j + \right.$$
$$\left.\lambda_k \frac{\partial h_k}{\partial\, {}_a\Delta_t^\alpha q_j} + \lambda_k \frac{\partial h_k}{\partial\, {}_t\Delta_b^\beta q_j}\right)\cdot \delta q_j^\sigma = 0 \tag{28}$$

Similarly, considering the arbitrariness of the value of δq_j^σ, we have

$$\frac{\partial \widetilde{T}}{\partial q_j^\sigma} + {}_t\Delta_{\rho(b)}^\alpha \frac{\partial \widetilde{T}}{\partial\, {}_a\Delta_t^\alpha q_j} + {}_a\Delta_t^\beta \frac{\partial \widetilde{T}}{\partial\, {}_t\Delta_b^\beta q_j} + Q_j +$$
$$\lambda_k \frac{\partial h_k}{\partial\, {}_a\Delta_t^\alpha q_j} + \lambda_k \frac{\partial h_k}{\partial\, {}_t\Delta_b^\beta q_j} = 0 \tag{29}$$

In Eq.(29), Q_j contains the conservative force Q'_j and the nonconservative force Q''_j. If Q'_j is potential, that is, there exists a function $V = V(t, q_i^\sigma)$ such that

$$Q'_j = -\frac{\partial V}{\partial q_j^\sigma} \tag{30}$$

Since

$$\frac{\partial V}{\partial\, {}_a\Delta_t^\alpha q_j} = 0, \frac{\partial V}{\partial\, {}_t\Delta_b^\beta q_j} = 0 \tag{31}$$

Substituting Eqs.(30) and (31) into Eq.(29), we have

$$\frac{\partial(\widetilde{T} - V)}{\partial q_j^\sigma} + {}_t\Delta_{\rho(b)}^\alpha \frac{\partial(\widetilde{T} - V)}{\partial\, {}_a\Delta_t^\alpha q_j} + {}_a\Delta_t^\beta \frac{\partial(\widetilde{T} - V)}{\partial\, {}_t\Delta_b^\beta q_j} +$$
$$\lambda_k \frac{\partial h_k}{\partial\, {}_a\Delta_t^\alpha q_j} + \lambda_k \frac{\partial h_k}{\partial\, {}_t\Delta_b^\beta q_j} + Q''_j = 0 \tag{32}$$

Let $L = \widetilde{T} - V$, Eq.(32) can be written as

$$\frac{\partial L}{\partial q_j^\sigma} + {}_t\Delta_{\rho(b)}^\alpha \frac{\partial L}{\partial\, {}_a\Delta_t^\alpha q_j} + {}_a\Delta_t^\beta \frac{\partial L}{\partial\, {}_t\Delta_b^\beta q_j} + \lambda_k \frac{\partial h_k}{\partial\, {}_a\Delta_t^\alpha q_j} +$$
$$\lambda_k \frac{\partial h_k}{\partial\, {}_t\Delta_b^\beta q_j} + Q''_j = 0 \tag{33}$$

If Q'_j has generalized potential, that is, there exists a function $U = U(t, q_i^\sigma, {}_a\Delta_t^\alpha q_i, {}_t\Delta_b^\beta q_i)$ such that

$$Q'_j = \frac{\partial U}{\partial q_j^\sigma} + {}_t\Delta_{\rho(b)}^\alpha \frac{\partial U}{\partial\, {}_a\Delta_t^\alpha q_j} + {}_a\Delta_t^\beta \frac{\partial U}{\partial\, {}_t\Delta_b^\beta q_j} \tag{34}$$

Substituting Eq.(34) into Eq.(29), we have

$$\frac{\partial(\widetilde{T} + U)}{\partial q_j^\sigma} + {}_t\Delta_{\rho(b)}^\alpha \frac{\partial(\widetilde{T} + U)}{\partial\, {}_a\Delta_t^\alpha q_j} + {}_a\Delta_t^\beta \frac{\partial(\widetilde{T} + U)}{\partial\, {}_t\Delta_b^\beta q_j} +$$
$$\lambda_k \frac{\partial h_k}{\partial\, {}_a\Delta_t^\alpha q_j} + \lambda_k \frac{\partial h_k}{\partial\, {}_t\Delta_b^\beta q_j} + Q''_j = 0 \tag{35}$$

Let $L = \widetilde{T} + U = \tilde{T} - V$, Eq.(35) can be written as

$$\frac{\partial L}{\partial q_j^\sigma} + {}_t\Delta_{\rho(b)}^\alpha \frac{\partial L}{\partial\, {}_a\Delta_t^\alpha q_j} + {}_a\Delta_t^\beta \frac{\partial L}{\partial\, {}_t\Delta_b^\beta q_j} + \lambda_k \frac{\partial h_k}{\partial\, {}_a\Delta_t^\alpha q_j} +$$
$$\lambda_k \frac{\partial h_k}{\partial\, {}_t\Delta_b^\beta q_j} + Q''_j = 0 \tag{36}$$

Eq.(36) is called discrete Lagrange equation with multipliers of the nonconservative system with dynamic constraint. From Eqs.(36) and (24), λ_k and q_i can be solved.

Remark 2 If $\alpha = 1$, L and h_k do not depend on ${}_t\Delta_b^\beta q_i$, and the discrete Lagrange equation of the nonconservative system with dynamic constraint can be obtained

$$\frac{\partial L(t, q_i(t+1), \Delta_d q_i)}{\partial q_j(t+1)} - \Delta_d \frac{\partial L}{\partial \Delta_d q_j} + \lambda_k \frac{\partial h_k}{\partial \Delta_d q_j} +$$
$$Q''_j = 0 \quad t \in \{a, a+1, \cdots, b-2\} \tag{37}$$

Eq.(37) is consistent with the result in Ref.[28].

4 Examples

Example 1 Consider the following nonconservative system

$$\begin{cases} L = \frac{1}{2}({}_a\Delta_t^\alpha q)^2 + q^\sigma\, {}_a\Delta_t^\alpha q \\ Q'' = -\,{}_a\Delta_t^\alpha q + {}_t\Delta_b^\beta q + q^\sigma \end{cases} \tag{38}$$

with dynamic constraint

$$h = ({}_a\Delta_t^\alpha q)^2 + q^\sigma = 0 \tag{39}$$

From Eq.(36), we have

$${}_t\Delta_{\rho(b)}^\alpha(q^\sigma + {}_a\Delta_t^\alpha q) + 2\lambda\, {}_a\Delta_t^\alpha q +$$

$$ {}_t\Delta_b^{\beta} q + q^{\sigma} = 0 \tag{40} $$

From Eq.(39), we have

$$ {}_t\Delta_{\rho(b)}^{\alpha} h = 2\, {}_a\Delta_t^{\alpha} q \cdot {}_t\Delta_{\rho(b)}^{\alpha} ({}_a\Delta_t^{\alpha} q) + {}_t\Delta_{\rho(b)}^{\alpha} q^{\sigma} = 0 \tag{41} $$

It follows from Eqs.(40) and (41) that

$$ \lambda = \frac{{}_t\Delta_{\rho(b)}^{\alpha} q^{\sigma}}{4({}_a\Delta_t^{\alpha} q)^2} - \frac{{}_t\Delta_{\rho(b)}^{\alpha} q^{\sigma} + {}_t\Delta_b^{\beta} q + q^{\sigma}}{2\, {}_a\Delta_t^{\alpha} q} \tag{42} $$

Specially, when $\alpha = \beta = 1$, we obtain

$$ \lambda = -\frac{\Delta_d q^{\sigma}}{4(\Delta_d q)^2} + \frac{\Delta_d q^{\sigma} + \Delta_d q - q^{\sigma}}{2\Delta_d q} \tag{43} $$

Example 2 There is a well-known example called Appell-Hamel[29]. We discuss the example of Appell-Hamel within fractional difference operators of Riemann-Liouville type.

The Lagrangian is

$$ L = \frac{1}{2} m [({}_a\Delta_t^{\alpha} q_1)^2 + ({}_a\Delta_t^{\alpha} q_2)^2 + ({}_a\Delta_t^{\alpha} q_3)^2] - mgq_3^{\sigma} \tag{44} $$

the dynamic constraint is

$$ h = \frac{b^2}{a^2} [({}_a\Delta_t^{\alpha} q_1)^2 + ({}_a\Delta_t^{\alpha} q_2)^2] - ({}_a\Delta_t^{\alpha} q_3)^2 = 0 \tag{45} $$

From Eq.(36), we have

$$ \begin{cases} m\, {}_t\Delta_{\rho(b)}^{\alpha}\, {}_a\Delta_t^{\alpha} q_1 + \lambda \dfrac{2b^2}{a^2}\, {}_a\Delta_t^{\alpha} q_1 = 0 \\ m\, {}_t\Delta_{\rho(b)}^{\alpha}\, {}_a\Delta_t^{\alpha} q_2 + \lambda \dfrac{2b^2}{a^2}\, {}_a\Delta_t^{\alpha} q_2 = 0 \\ -mg + m\, {}_t\Delta_{\rho(b)}^{\alpha}\, {}_a\Delta_t^{\alpha} q_3 - 2\lambda\, {}_a\Delta_t^{\alpha} q_3 = 0 \end{cases} \tag{46} $$

From Eq.(45), we have

$$ {}_t\Delta_{\rho(b)}^{\alpha} h = \frac{2b^2}{a^2}\, {}_a\Delta_t^{\alpha} q_1 \cdot {}_t\Delta_{\rho(b)}^{\alpha} ({}_a\Delta_t^{\alpha} q_1) + \frac{2b^2}{a^2}\, {}_a\Delta_t^{\alpha} q_2 \cdot {}_t\Delta_{\rho(b)}^{\alpha} ({}_a\Delta_t^{\alpha} q_2) - 2\, {}_a\Delta_t^{\alpha} q_3 \cdot {}_t\Delta_{\rho(b)}^{\alpha} ({}_a\Delta_t^{\alpha} q_3) = 0 \tag{47} $$

It follows from Eqs.(46) and (47) that

$$ \lambda = \frac{-a^4 g\, {}_a\Delta_t^{\alpha} q_3}{2b^4 [({}_a\Delta_t^{\alpha} q_1)^2 + ({}_a\Delta_t^{\alpha} q_2)^2] + 2a^4 ({}_a\Delta_t^{\alpha} q_3)^2} \tag{48} $$

Specially, when $\alpha = \beta = 1$, we obtain

$$ \lambda = \frac{-a^4 g \Delta_d q_3}{2b^4 [(\Delta_d q_1)^2 + (\Delta_d q_2)^2] + 2a^4 (\Delta_d q_3)^2} \tag{49} $$

5 Conclusions

Using the properties of the time scale calculus, discrete Lagrange equations of the nonconservative system and the nonconservative system with dynamic constraint in terms of fractional difference operators of Riemann-Liouville type are obtained. Two special cases are given. In addition, the proposed method can also be applied to study other mechanical systems, such as the Hamiltonian system and the Birkhoffian system.

In addition, we will conduct further research in symmetry and conserved quantity, perturbation to symmetry and adiabatic invariants within fractional difference operators of Riemann-Liouville type of constrained mechanical systems.

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