Adiabatic Invariant for Dynamic Systems on Time Scale

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Abstract: Perturbation to Noether symmetry and adiabatic invariants for Birkhoffian system, Hamiltonian system and Lagrangian system with delta derivative are investigated, respectively. Firstly, the definition and some related properties of time scale calculus are listed simply as preliminaries. Secondly, the Birkhoffian system with delta derivative is studied. Based on the differential equation of motion as well as Noether symmetry and conserved quantity, perturbation to Noether symmetry and adiabatic invariant are investigated. Thirdly, adiabatic invariants for the Hamiltonian system and the Lagrangian system are presented through some transformations. And finally, an example is given to illustrate the methods and results.

Key words: perturbation to Noether symmetry; adiabatic invariant; dynamic system; time scale

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0 Introduction

Adiabatic invariant means a physical quantity, which varies more slowly than the changing parameter. The adiabatic invariant is closely related to the integrable property of the mechanic system. Because one of the important tasks of analytical mechanics is to find the solutions of the differential equations of motion, it is significant to study the adiabatic invariant. The adiabatic invariant was first introduced by Burgers in 1917[1]. Since then, many results about perturbation to symmetry and adiabatic invariant have been obtained[2-7].

Time scale, which means an arbitrary nonempty closed subset of the real number set, was first proposed by Stefan Hilger in 1988[8]. Because the integer number set Z and the real number set R are both examples of time scale, the theory of time scale can harmonize the continuous analysis and the discrete analysis. A lot of results for dynamic system on time scale have been obtained, for example, dynamic equations on time scale[9-10], calculus of variations on time scale[11-12], Noether symmetry and conserved quantity on time scale[13-16], and so on. There are two derivatives used commonly on time scale, namely delta derivative and nabla derivative. In this paper, the adiabatic invariant for dynamic systems with delta derivative is only presented.

1 Preliminaries

Here some properties and definitions about time scale calculus are reviewed[9].

Let T be a time scale, the forward jump operator $\sigma: T \rightarrow T$ and the graininess function θ: $T \rightarrow [0, \infty)$ are defined as

$$\sigma(t) = \inf\{p \in T: p > t\}, \theta(t) = \sigma(t) - t,$$

where $t \in T$, $\inf\varnothing = \sup T$, $\sup\varnothing = \inf T$.

Let $f: T \rightarrow \mathrm{R}$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\omega \in (t-\delta, t+\delta) \cap T$, $\left| f(\sigma(t)) - f(\omega) - f^{\Delta}(t)[\sigma(t) - \omega] \right| \leqslant \varepsilon \left| \sigma(t) - \omega \right|$ holds, then $f^{\Delta}(t)$ is called delta derivative of f at point t. $f^{\Delta}(t)$ can also be denoted as $\frac{\Delta}{\Delta t} f(t)$.

Remark 1 When $T = \mathrm{R}$, $\sigma(t) = t$, $\theta(t) = 0$, $f^{\Delta}(t) = f'(t)$. When $T = \mathrm{Z}$, $\sigma(t) = t+1$, $\theta(t) = 1$, $f^{\Delta}(t) = \Delta_{\mathrm{d}} f = f(t+1) - f(t)$.

Supposing that $f, g: T \rightarrow \mathrm{R}$ are differentiable,

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then for any constants α and β, we have

$$f(\sigma(t))=f^{\sigma}(t)=f(t)+\theta(t)f^{\Delta}(t) \tag{1}$$

$$(\alpha f+\beta g)^{\Delta}(t)=\alpha f^{\Delta}(t)+\beta g^{\Delta}(t) \tag{2}$$

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g^{\sigma}(t)+f(t)g^{\Delta}(t) \tag{3}$$

A function $f: T\rightarrow \mathrm{R}$ is called rd-continuous if it is continuous at right-dense points and its left-sided limits exist at left-dense points. The set of rd-continuous functions is denoted by C_{rd}. C_{rd}^{1} denotes the set of these functions whose derivatives are still rd-continuous.

A function $F: T\rightarrow \mathrm{R}$ is called an antiderivative of $f: T\rightarrow \mathrm{R}$ provided $F^{\Delta}(t)=f(t)$ holds.

It is noted that every rd-continuous function has an antiderivative. In particular, if $t_0\in T$, then F defined by $F(t)=\int_{t_0}^{t}f(\tau)\Delta\tau$ for $t\in T$ is an antiderivative of f.

2 Adiabatic Invariant for Birkhoffian System on Time Scale

2.1 Differential equation of motion

From the following functional

$$I(a_{\mu}(\cdot))=[R_{\nu}(t,a_{\mu}^{\sigma})a_{\nu}^{\Delta}-B(t,a_{\mu}^{\sigma})]\Delta t\rightarrow \mathrm{ext}$$
$$\delta a_{\nu}\Big|_{t=t_1}=\delta a_{\nu}\Big|_{t=t_2}=0 \tag{4}$$

the Birkhoff equation on time scale[15] is given as

$$\frac{\partial R_{v}(t,a_{\mu}^{\sigma})}{\partial a_{\rho}^{\sigma}}\cdot a_{v}^{\Delta}-\frac{\partial B(t,a_{\mu}^{\sigma})}{\partial a_{\rho}^{\sigma}}-R_{\rho}^{\Delta}=0 \tag{5}$$

where $a_{\mu}^{\sigma}(t)=(a_{\mu}\circ\sigma)(t)$, $a_{\nu}^{\Delta}(t)$ is the delta derivative, $t\in T$; the Birkhoffian $B:\mathrm{R}\times\mathrm{R}^{2n}\rightarrow\mathrm{R}$ and the Birkhoff's functions $R_{v}:\mathrm{R}\times\mathrm{R}^{2n}\rightarrow\mathrm{R}$ are assumed to be $\mathrm{C}_{\mathrm{rd}}^{1}$, $\mu,\nu,\rho=1,2,\cdots,2n$; $\delta(\cdot)$ denotes the isochronous variation.

2.2 Noether symmetry and conserved quantity

Definition 1 Quantity C is said to be a conserved quantity if and only if $\frac{\Delta}{\Delta t}C=0$ along the differential equation of motion on time scale.

Theorem 1[15] If $I(a_{\mu}(\cdot))$ in Eq.(4) is invariant under the infinitesimal transformations

$$t^{*}=t,a_{\mu}^{*}=a_{\mu}+\varepsilon\xi_{B\mu}^{0}(t,a_{\nu}) \tag{6}$$

that is, for any $[t_a,t_b]\subseteq[t_1,t_2]$,

$$\int_{t_a}^{t_b}[R_{\nu}(t,a_{\mu}^{\sigma})a_{\nu}^{\Delta}-B(t,a_{\mu}^{\sigma})]\Delta t=$$
$$\int_{t_a}^{t_b}[R_{\nu}(t,a_{\mu}^{*\sigma})a_{\nu}^{*\Delta}-B(t,a_{\mu}^{*\sigma})]\Delta t \tag{7}$$

holds, then a conserved quantity exists as

$$C_{B0}=R_{\mu}(t,a_{\nu}^{\sigma})\xi_{B\mu}^{0}(t,a_{\nu}) \tag{8}$$

where ε is an infinitesimal parameter, and $\xi_{B\mu}^{0}$ the infinitesimal generator of the infinitesimal transformation (Eq.(6)).

Theorem 2 [15]If $I(a_{\mu}(\cdot))$ in Eq.(4) is invariant under the infinitesimal transformations

$$t^{*}=t+\varepsilon\xi_{B0}^{0}(t,a_{\nu}),a_{\mu}^{*}=a_{\mu}+\varepsilon\xi_{B\mu}^{0}(t,a_{\nu}) \tag{9}$$

that is, for any $[t_a,t_b]\subseteq[t_1,t_2]$,

$$\int_{t_a}^{t_b}[R_{\nu}(t,a_{\mu}^{\sigma})a_{\nu}^{\Delta}-B(t,a_{\mu}^{\sigma})]\Delta t=$$
$$\int_{t_a^{*}}^{t_b^{*}}[R_{\nu}(t^{*},a_{\mu}^{*\sigma^{*}}(t^{*}))a_{\nu}^{*\Delta^{*}}(t^{*})-B(t^{*},a_{\mu}^{*\sigma^{*}}(t^{*}))]\Delta^{*}t^{*} \tag{10}$$

holds, then the conserved quantity can be obtained as

$$C_{B}=R_{\mu}\xi_{B\mu}^{0}-\left[\theta(t)\left(\frac{\partial R_{\nu}}{\partial t}a_{\nu}^{\Delta}-\frac{\partial B}{\partial t}\right)+B\right]\xi_{B0}^{0} \tag{11}$$

where ε is an infinitesimal parameter, ξ_{B0}^{0} and $\xi_{B\mu}^{0}$ are the infinitesimal generators of the infinitesimal transformation (Eq.(9)).

2.3 Perturbation to Noether symmetry and adiabatic invariant

Assume that the Birkhoffian system (Eq.(5)) is disturbed by small forces $\varepsilon W_{B\rho}(t,a_{\mu}^{\sigma})$, $\rho=1,2,\cdots,2n$, then we have

$$\frac{\partial R_{v}(t,a_{\mu}^{\sigma})}{\partial a_{\rho}^{\sigma}}\cdot a_{v}^{\Delta}-\frac{\partial B(t,a_{\mu}^{\sigma})}{\partial a_{\rho}^{\sigma}}-R_{\rho}^{\Delta}=$$
$$\varepsilon W_{B\rho}(t,a_{\mu}^{\sigma}) \tag{12}$$

Under the small disturbance, conserved quantity may change. It is supposed that this change is a small perturbation on the basis of the undisturbed system. Therefore, the infinitesimal generators of the disturbed system can be expressed as

$$\xi_{B0}=\xi_{B0}^{0}+\varepsilon\xi_{B0}^{1}+\varepsilon^{2}\xi_{B0}^{2}+\cdots$$
$$\xi_{B\mu}=\xi_{B\mu}^{0}+\varepsilon\xi_{B\mu}^{1}+\varepsilon^{2}\xi_{B\mu}^{2}+\cdots \tag{13}$$

Definition 2 If a quantity I_z satisfies that the highest power of ε is z and $\Delta I_z/\Delta t$ is in direct proportion to ε^{z+1}, then I_z is a z-th order adiabatic invariant on time scale.

Theorem 3 Under the infinitesimal transformations

$$t^{*}=t,a_{\mu}^{*}=a_{\mu}+\varepsilon\xi_{B\mu}(t,a_{\nu}) \tag{14}$$

when the Birkhoffian system (Eq.(5)) is disturbed, if the infinitesimal generator $\xi_{B\rho}^{j}, j=0,1,2,\cdots,z$ satisfies

$$\frac{\partial R_{\nu}(t,a_{\mu}^{\sigma})}{\partial a_{\rho}^{\sigma}}\xi_{B\rho}^{j\sigma}a_{\nu}^{\Delta}+R_{\nu}(t,a_{\mu}^{\sigma})\xi_{B\nu}^{j\Delta}-\frac{\partial B(t,a_{\mu}^{\sigma})}{\partial a_{\rho}^{\sigma}}\xi_{B\rho}^{j\sigma}-W_{B\rho}\xi_{B\rho}^{(j-1)\sigma}=0 \tag{15}$$

then a z-th order adiabatic invariant exists as

$$I_{Bz0}=\varepsilon^{j}R_{\mu}(t,a_{\nu}^{\sigma})\xi_{B\mu}^{j}(t,a_{\nu}) \tag{16}$$

where $\xi_{B\mu}^{j-1}=0$ when $j=0$.

Proof From Eqs. (12,15) we can get

$$\frac{\Delta}{\Delta t}I_{Bz0}=\sum_{j=0}^{z}\varepsilon^{j}\frac{\Delta}{\Delta t}\Big[R_{\mu}(t,a_{\nu}^{\sigma})\xi_{B\mu}^{j}\big(t,a_{\nu}\big)\Big]=\sum_{j=0}^{z}\varepsilon^{j}\Big(R_{\mu}\xi_{B\mu}^{j\Delta}+R_{\mu}^{\Delta}\xi_{B\mu}^{j\sigma}\Big)=$$

$$\sum_{j=0}^{z}\varepsilon^{j}\left[-\frac{\partial R_{\nu}\big(t,a_{\mu}^{\sigma}\big)}{\partial a_{\rho}^{\sigma}}\xi_{B\rho}^{j\sigma}a_{\nu}^{\Delta}+\frac{\partial B\big(t,a_{\mu}^{\sigma}\big)}{\partial a_{\rho}^{\sigma}}\xi_{B\rho}^{j\sigma}+W_{B\rho}\xi_{B\rho}^{(j-1)\sigma}+R_{\mu}^{\Delta}\xi_{B\mu}^{j\sigma}\right]=\sum_{j=0}^{z}\varepsilon^{j}\Big[-\varepsilon W_{B\rho}\big(t,a_{\mu}^{\sigma}\big)\xi_{B\rho}^{j\sigma}+W_{B\rho}\xi_{B\rho}^{(j-1)\sigma}\Big]=$$

$$-\varepsilon^{z+1}W_{B\rho}\big(t,a_{\mu}^{\sigma}\big)\xi_{B\rho}^{z\sigma}.$$

This proof is completed.

Theorem 4 Under the infinitesimal transformations

$$t^{*}=t+\varepsilon\xi_{B0}\big(t,a_{\nu}\big),a_{\mu}^{*}=a_{\mu}+\varepsilon\xi_{B\mu}\big(t,a_{\nu}\big) \tag{17}$$

when the Birkhoffian system (Eq.(5)) is disturbed, a z-th order adiabatic invariant exists as

$$I_{Bz}=\varepsilon^{j}\left\{R_{\mu}\xi_{B\mu}^{j}-\left[\theta(t)\left(\frac{\partial R_{\nu}}{\partial t}a_{\nu}^{\Delta}-\frac{\partial B}{\partial t}\right)+B\right]\xi_{B0}^{j}\right\} \tag{18}$$

Proof Consider

$$I(a_{\mu}(\cdot))=\int_{t_a}^{t_b}[R_{\nu}(t,a_{\mu}^{\sigma})a_{\nu}^{\Delta}-B(t,a_{\mu}^{\sigma})]\Delta t=\int_{t_a}^{t_b}G(t,a_{\mu}^{\sigma},a_{\mu}^{\Delta})\Delta t \tag{19}$$

$$\bar{I}\big(s(\cdot),a_{\mu}(\cdot)\big)=\int_{t_a}^{t_b}\left[R_{\nu}(s^{\sigma}-\theta(t)s^{\Delta},a_{\mu}^{\sigma})\frac{a_{\nu}^{\Delta}}{s^{\Delta}}-B(s^{\sigma}-\theta(t)s^{\Delta},a_{\mu}^{\sigma})\right]s^{\Delta}\Delta t=\int_{t_a}^{t_b}\bar{G}(t,s^{\sigma},a_{\mu}^{\sigma},s^{\Delta},a_{\mu}^{\Delta})\Delta t \tag{20}$$

When $s(t)=t$, we have

$$\int_{t_a}^{t_b}[R_{\nu}(t,a_{\mu}^{\sigma})a_{\nu}^{\Delta}-B(t,a_{\mu}^{\sigma})]\Delta t=\int_{t_a}^{t_b}\left[R_{\nu}(s^{\sigma}-\theta(t)s^{\Delta},a_{\mu}^{\sigma})\frac{a_{\nu}^{\Delta}}{s^{\Delta}}-B(s^{\sigma}-\theta(t)s^{\Delta},a_{\mu}^{\sigma})\right]s^{\Delta}\Delta t \tag{21}$$

i.e.

$$G(t,a_{\mu}^{\sigma},a_{\mu}^{\Delta})=\bar{G}(t,s^{\sigma},a_{\mu}^{\sigma},s^{\Delta},a_{\mu}^{\Delta}) \tag{22}$$

Making use of Theorem 3, we obtain that when $s(t)=t$

$$I_{Bz}=\sum_{j=0}^{z}\varepsilon^{j}\left(\frac{\partial\bar{G}}{\partial s^{\Delta}}\cdot\xi_{B0}^{j}+\frac{\partial\bar{G}}{\partial a_{\nu}^{\Delta}}\cdot\xi_{B\nu}^{j}\right) \tag{23}$$

where

$$\frac{\partial\bar{G}\big(t,s^{\sigma},a_{\mu}^{\sigma},s^{\Delta},a_{\mu}^{\Delta}\big)}{\partial a_{\nu}^{\Delta}}=R_{\nu}(s^{\sigma}-\theta(t)s^{\Delta},a_{\mu}^{\sigma})=R_{\nu}(t,a_{\mu}^{\sigma}) \tag{24}$$

$$\frac{\partial\bar{G}(t,s^{\sigma},a_{\mu}^{\sigma},s^{\Delta},a_{\mu}^{\Delta})}{\partial s^{\Delta}}=-\theta(t)\partial_1R_{\nu}(s^{\sigma}-\theta(t)s^{\Delta},a_{\mu}^{\sigma})\cdot a_{\nu}^{\Delta}+\theta(t)\partial_1B(s^{\sigma}-\theta(t)s^{\Delta},a_{\mu}^{\sigma})\cdot s^{\Delta}-B(s^{\sigma}-\theta(t)s^{\Delta},a_{\mu}^{\sigma})=-\theta(t)\partial_1R_{\nu}(t,a_{\mu}^{\sigma})\cdot a_{\nu}^{\Delta}(t)+\theta(t)\partial_1B(t,a_{\mu}^{\sigma})-B(t,a_{\mu}^{\sigma})=-\theta(t)\frac{\partial R_{\nu}(t,a_{\mu}^{\sigma})}{\partial t}\cdot a_{\nu}^{\Delta}(t)+\theta(t)\frac{\partial B(t,a_{\mu}^{\sigma})}{\partial t}-B(t,a_{\mu}^{\sigma}) \tag{25}$$

where ∂_1B and ∂_1R_{ν} denote the partial derivative of B and R_{ν} with respect to their first variables, respectively. Substituting Eqs. (24, 25) into Eq.(23), the intended result can be obtained.

This proof is completed.

Remark 2 Theorem 3 and Theorem 4 reduce to Theorem 1 and Theorem 2 when $z=0$ respectively. That is, when the Birkhoffian system is undisturbed, a conserved quantity can be got. When the Birkhoffian system is disturbed, a z-th order adiabatic invariant can be got.

Remark 3 When $T=\mathbf{R}$, from Eq.(18) we have

$$I_{Bzc}=\sum_{j=0}^{z}\varepsilon^{j}(R_{\mu}\xi_{B\mu}^{j}-B\xi_{B0}^{j}) \tag{26}$$

Eq.(26) is the classical adiabatic invariant for the classical Birkhoffian system[17].

Remark 4 When $T=\mathbf{Z}$, from Eq.(18) we have

$$I_{Bzd}=\varepsilon^{j}\left\{R_{\mu}(t,a_{\nu}(t+1))\xi_{B\mu}^{j}(t,a_{\nu}(t))-\left[\theta(t)\left(\frac{\partial R_{v}(t,a_{\mu}(t+1))}{\partial t}\Delta_{\rm d}a_{v}(t)-\frac{\partial B}{\partial t}\right)+B(t,a_{\mu}(t+1))\right]\xi_{B0}^{j}(t,a_{\nu}(t))\right\} \tag{27}$$

Eq. (27) is the adiabatic invariant for the discrete Birkhoffian system.

Considering that T has many other special values, Theorem 3 and Theorem 4 have the universal significance.

3 Adiabatic Invariants for Hamiltonian System and Lagrangian System on Time Scale

3.1 Adiabatic invariant for Hamiltonian system

From the following transformations

$$a_\mu^\sigma = \begin{cases} q_\mu^\sigma & \mu = 1,2,\cdots,n \\ p_{\mu-n} & \mu = n+1, n+2, \cdots, 2n \end{cases}$$

$$R_\mu = \begin{cases} p_\mu & \mu = 1,2,\cdots,n \\ 0 & \mu = n+1, n+2, \cdots, 2n \end{cases}$$

$$B = H \tag{28}$$

disturbed Hamilton equation can be obtained from Eq. (12), that is

$$p_i^\Delta = -\frac{\partial H(t, q_m^\sigma, p_m)}{\partial q_i^\sigma} - \varepsilon W_{Hi}(t, q_m^\sigma, p_m)$$

$$q_i^\Delta = \frac{\partial H(t, q_m^\sigma, p_m)}{\partial p_i} \quad i, m = 1, 2, \cdots, n \tag{29}$$

Theorem 5 For the disturbed Hamiltonian system (Eq.(29)), if the infinitesimal generator ξ_{Hi}^j, $j = 0, 1, 2, \cdots, z$ satisfies

$$-\frac{\partial H(t, q_m^\sigma, p_m)}{\partial q_i^\sigma} \xi_{Hi}^{j\sigma}(t, q_m, p_m) + p_i \xi_{Hi}^{j\Delta} - W_{Hi} \xi_{Hi}^{(j-1)\sigma} = 0 \tag{30}$$

then a z-th order adiabatic invariant exists as

$$I_{Hz0} = \sum_{j=0}^{z} \varepsilon^j p_i \xi_{Hi}^j(t, q_m, p_m) \tag{31}$$

where $\xi_{Hi}^{j-1} = 0$ when $j = 0$.

Proof From Eqs. (29, 30), we get

$$\frac{\Delta}{\Delta t} I_{Hz0} = \sum_{j=0}^{z} \varepsilon^j \frac{\Delta}{\Delta t}(p_i \xi_{Hi}^j) =$$

$$\sum_{j=0}^{z} \varepsilon^j (p_i \xi_{Hi}^{j\Delta} + p_i^\Delta \xi_{Hi}^{j\sigma}) =$$

$$\sum_{j=0}^{z} \varepsilon^j \left[\frac{\partial H(t, q_m^\sigma, p_m)}{\partial q_i^\sigma} \xi_{Hi}^{j\sigma} + W_{Hi} \xi_{Hi}^{(j-1)\sigma} + p_i^\Delta \xi_{Hi}^{j\sigma} \right] =$$

$$\sum_{j=0}^{z} \varepsilon^j [-\varepsilon W_{Hi}(t, q_m^\sigma, p_m) \xi_{Hi}^{j\sigma} + W_{Hi} \xi_{Hi}^{(j-1)\sigma}] =$$

$$-\varepsilon^{z+1} W_{Hi}(t, q_m^\sigma, p_m) \xi_{Hi}^{z\sigma}$$

This proof is completed.

Theorem 6 Under the infinitesimal transformations with the time and the coordinates both changing, the disturbed Hamiltonian system (Eq. (29)) has an adiabatic invariant

$$I_{Hz} = \sum_{j=0}^{z} \varepsilon^j \left\{ p_i \xi_{Hi}^j + \left[\frac{\partial H}{\partial t} \theta(t) - H \right] \xi_{H0}^j \right\} \tag{32}$$

Remark 5 Adiabatic invariant for the Hamiltonian system can be obtained through transformation (Eq. (28)), and they can also be obtained through the similar method used for the Birkhoffian system.

Remark 6 From Remark 2, put $z = 0$ in Eqs. (31, 32), conserved quantities for the undisturbed Hamiltonian system can be got, which are consistent with the results in Refs. [16,18].

3.2 Adiabatic invariant for Lagrangian system

From the following transformations

$$p_i = \frac{\partial L(t, q_m^\sigma, q_m^\Delta)}{\partial q_i^\Delta}, H = p_i q_i^\Delta - L(t, q_m^\sigma, q_m^\Delta) \tag{33}$$

disturbed Lagrange equation can be obtained from Eq. (29), that is

$$\frac{\Delta}{\Delta t} \frac{\partial L(t, q_m^\sigma, q_m^\Delta)}{\partial q_i^\Delta} = \frac{\partial L(t, q_m^\sigma, q_m^\Delta)}{\partial q_i^\sigma} - \varepsilon W_{Li}(t, q_m^\sigma, q_m^\Delta) \tag{34}$$

Theorem 7 For the disturbed Lagrangian system (Eq. (34)), if the infinitesimal generator ξ_{Li}^j, $j = 0, 1, 2, \cdots, z$ satisfies

$$\frac{\partial L(t, q_m^\sigma, q_m^\Delta)}{\partial q_i^\sigma} \xi_{Li}^{j\sigma}(t, q_m) + \frac{\partial L(t, q_m^\sigma, q_m^\Delta)}{\partial q_i^\Delta} \xi_{Li}^{j\Delta} - W_{Li} \xi_{Li}^{(j-1)\sigma} = 0 \tag{35}$$

then a z-th order adiabatic invariant exists as

$$I_{Lz0} = \sum_{j=0}^{z} \varepsilon^j \frac{\partial L(t, q_m^\sigma, q_m^\Delta)}{\partial q_i^\Delta} \xi_{Li}^j(t, q_m) \tag{36}$$

where $\xi_{Li}^{j-1} = 0$ when $j = 0$.

Proof From Eqs. (34,35), we can get

$$\frac{\Delta}{\Delta t} I_{Lz0} = \sum_{j=0}^{z} \varepsilon^j \frac{\Delta}{\Delta t} \left[\frac{\partial L(t, q_m^\sigma, q_m^\Delta)}{\partial q_i^\Delta} \xi_{Li}^j \right] =$$

$$\sum_{j=0}^{z} \varepsilon^j \left[\frac{\partial L(t, q_m^\sigma, q_m^\Delta)}{\partial q_i^\Delta} \xi_{Li}^{j\Delta} + \frac{\Delta}{\Delta t} \frac{\partial L}{\partial q_i^\Delta} \xi_{Li}^{j\sigma} \right] =$$

$$\sum_{j=0}^{z} \varepsilon^j \left[-\frac{\partial L(t, q_m^\sigma, q_m^\Delta)}{\partial q_i^\sigma} \xi_{Li}^{j\sigma} + W_{Li} \xi_{Li}^{(j-1)\sigma} + \right.$$

$$\left.\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_i^\Delta}\xi_{Li}^{j\sigma}\right]=\sum_{j=0}^{z}\varepsilon^j[-\varepsilon W_{Li}(t,q_m^\sigma,q_m^\Delta)\xi_{Li}^{j\sigma}+ W_{Li}\xi_{Li}^{(j-1)\sigma}]=-\varepsilon^{z+1}W_{Li}(t,q_m^\sigma,q_m^\Delta)\xi_{Li}^{z\sigma}$$

This proof is completed.

Theorem 8 Under the infinitesimal transformations with the time and the coordinates both changing, the disturbed Lagrangian system (Eq. (34)) has an adiabatic invariant

$$I_{Lz}=\sum_{j=0}^{z}\varepsilon^j\left\{\frac{\partial L(t,q_m^\sigma,q_m^\Delta)}{\partial q_i^\Delta}\xi_{Li}^j+\left[L-\frac{\partial L}{\partial q_i^\Delta}q_i^\Delta-\frac{\partial L}{\partial t}\theta(t)\right]\xi_{L0}^j\right\} \tag{37}$$

Remark 7 Adiabatic invariant for the Lagrangian system can be obtained through transformation (Eq. (33)), and they can also be achieved through the similar method used for the Birkhoffian system.

Remark 8 The results of the adiabatic invariant for Lagrangian system are consistent with those obtained in Ref. [19].

Remark 9 From Remark 2, put $z=0$ in Eq. (37), a conserved quantity for the undisturbed Lagrangian system can be got, which is consistent with the result in Ref. [13].

4 Illustrative Example

Try to find out the adiabatic invariant for the following Birkhoffian system

$$B=t^2a_1^\sigma a_2^\sigma, R_1=a_2^\sigma, R_2=0 \tag{38}$$

on the time scale $T=\left\{2^n: n\in \mathrm{N}\cup\{0\}\right\}$.

From the given time scale, we get $\sigma(t)=2t$, $\theta(t)=t$.

When the system is undisturbed, that is, let $j=0$ in Eq. (15), we get

$$-t^2a_2^\sigma\xi_1^{0\sigma}+(a_1^\Delta-t^2a_1^\sigma)\xi_2^{0\sigma}+a_2^\sigma\xi_1^{0\Delta}=0 \tag{39}$$

Taking calculation, we have

$$\xi_1^0=a_1, \xi_2^0=-a_2 \tag{40}$$

Theorem 3 gives

$$I_{B00}=a_1a_2^\sigma \tag{41}$$

When the system is disturbed by

$$\varepsilon W_1=0, \varepsilon W_2=\varepsilon(2t^3-1) \tag{42}$$

Let $j=1$ in Eq. (15), we get

$$-t^2a_2^\sigma\xi_1^{1\sigma}+(a_1^\Delta-t^2a_1^\sigma)\xi_2^{1\sigma}+a_2^\sigma\xi_1^{1\Delta}-a_1^\sigma W_1+a_2^\sigma W_2=0 \tag{43}$$

Taking calculation, we have

$$\xi_1^1=t, \xi_2^1=0 \tag{44}$$

Theorem 3 gives

$$I_{B10}=a_1a_2^\sigma+\varepsilon ta_2^\sigma \tag{45}$$

Furthermore, the higher order adiabatic invariant can be got.

5 Conclusions

Perturbation to Noether symmetry and adiabatic invariant for dynamic systems on time scale are presented in this paper. Theorem 3-Theorem 6 are new work. Theorem 1, Theorem 2, Theorem 7 and Theorem 8 are consistent with the existing results. Several special cases are given in the form of Remark simply.

As mentioned in Introduction, nabla derivative and delta derivative have equal importance for dynamic system. In fact, adiabatic invariant for dynamic system with nabla derivative can be obtained through the similar method used in this paper. Besides, duality principle is an important and elegant method used on time scale. Therefore, adiabatic invariant for dynamic system with nabla derivative can also be studied through duality principle, which is not mentioned here.

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