Recent Advances on Herglotz's Generalized Variational Principle of Nonconservative Dynamics

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Abstract: This paper summarized the recent development on Herglotz's generalized variational principle and its symmetries and conserved quantities for nonconservative dynamical systems. Taking Lagrangian mechanics, Hamiltonian mechanics and Birkhoffian mechanics as three research frames, we introduce Herglotz's generalized variational principle, dynamical equations of Herglotz type, Noether symmetry and conserved quantities, and their generalization to time-delay dynamics, fractional dynamics and time-scale dynamics, and put forward some problems as suggestions for future research.

Key words: nonconservative dynamics; Herglotz's generalized variational principle; Lagrangian mechanics; Hamiltonian mechanics; Birkhoffian mechanics

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0 Introduction

The variational principle of mechanics is, as known to all, the smallest and most concise logical starting point that hides the result of the whole mechanics discipline. According to this kind of principle, the actual motion of a mechanical system always minimizes or maximizes a function or a functional than any possible motion in its vicinity. For example, the famous Hamilton principle can be stated as that in a conservative mechanical system with ideal, bilateral and holonomic constraints, the action functional of a real motion has an extreme value in all possible motions from one initial position to another known position within the same time period, i.e.

$$\delta \int_{t_0}^{t_1} L(t, q_s, \dot{q}_s) \mathrm{d}t = 0 \tag{1}$$

However, for holonomic nonconservative systems, or nonholonomic systems, the principle (1) no longer holds. For a holonomic nonconservative system, the Hamilton principle is extended as follows

$$\int_{t_0}^{t_1} (\delta T + Q_s \delta q_s) \mathrm{d}t = 0 \tag{2}$$

In general, the principle (2) is not a stable action principle because it cannot be expressed as an extreme value of a function or functional.

Herglotz proposed a class of generalized variational principles when he studied contact transformation and its relation to Hamilton system and Poisson bracket, see Herglotz[1-2] and Guenther et al.[3], that is[4]:

Let the functional $z = z[x; s]$ of $x(t)$ be given by the following differential equation

$$\dot{z} = L(t, x(t), \dot{x}(t), z(t)), \quad t \in [0, s] \tag{3}$$

and let the functions $\eta = (\eta^1(t), \cdots, \eta^n(t))$ have continuous first derivatives and satisfy the boundary conditions $\eta(0) = \eta(s) = 0$ but otherwise be arbitrary. Then the value of the functional $z = z[x; s]$ is an extremum for functions $x(t)$ that satisfies the condition

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} z[x + \varepsilon\eta; s] \bigg|_{\varepsilon = 0} = 0 \tag{4}$$

Compared with the classical variational princi-

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ple, Herglotz's generalized variational principle has several characteristics. The first is that it gives a variational description of nonconservative dynamical processes while the classical variational principle cannot express a nonconservative system as an extreme value of a functional. The second is that when L does not contain $z(t)$ explicitly, $z(t_1)$ becomes the classical Hamilton action, and the principle is reduced to the Hamilton principle of the holonomic conservative system. Therefore, Herglotz's generalized variational principle can describe all the physical processes that can be described by the classical variational principle, and the problem that the classical variational principle cannot be applied. The third is that Herglotz's generalized variational principle unifies both conservative and nonconservative processes into the same dynamical model, and thus can systematically deal with actual dynamical problems. The fourth is that the generalized variational problems of Herglotz type can be extended to multivariable dynamics problems and infinite dimensional dynamics systems, etc.

In 2001, under the guidance of Guenther, Dr. Georgieva of Oregon State University proposed and studied Herglotz's generalized variational principle and Noether's theorem in his doctoral dissertation[4], and obtained some preliminary results[4-9]. Donchev[10] used Herglotz's generalized variational principle to give variational descriptions of Böcher equation, nonlinear Schrödinger equation and other mathematical physical equations, none of which had variational descriptions under the classical variational principle. In recent years, Herglotz's generalized variational principle and its application have attracted a lot of attention. In this paper, the dynamics, symmetry and conserved quantities of nonconservative systems based on Herglotz's generalized variational principle are reviewed in detail.

1 Lagrangian Mechanics of Herglotz Type

1.1 Herglotz's generalized variational principle

The variational problem of Herglotz type for a nonconservative Lagrange system can be formulated as follows:

Determine the trajectories $q_s(t)$ satisfying given endpoint conditions $q_s(t)\big|_{t=t_0}=q_{s0}$, $q_s(t)\big|_{t=t_1}=q_{s1}$ $(s=1,2,\cdots,n)$ that extremize the value $z(t_1)\rightarrow \text{extr}$, where the functional z is defined by the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t}=L(t,q_s(t),\dot{q}_s(t),z(t)),t\in[t_0,t_1] \tag{5}$$

and subjected to the initial condition $z(t)\big|_{t=t_0}=z_0$. Here, $L(t,q_s(t),\dot{q}_s(t),z(t))$ can be called the Lagrangian in the sense of Herglotz, $q_s(t)$ are the generalized coordinates, q_{s0}, q_{s1}, and z_0 are fixed real constants.

Functional z is called the Hamilton-Herglotz action. The variational problem above can be referred to as Herglotz's generalized variational principle of nonconservative Lagrangian mechanics.

If L does not contain $z(t)$, then the functional z becomes the classical Hamilton action $z=\int_{t_0}^{t_1}L(t,q_s(t),\dot{q}_s(t))\mathrm{d}t$, and the principle is reduced to the classical Hamilton principle $\delta z=0$ for holonomic conservative systems.

1.2 Euler-Lagrange equations of Herglotz type

Calculating the isochronous variation of Eq.(5), we have

$$\delta\dot{z}=\frac{\partial L}{\partial q_s}\delta q_s+\frac{\partial L}{\partial \dot{q}_s}\delta\dot{q}_s+\frac{\partial L}{\partial z}\delta z \tag{6}$$

The solution of Eq.(6) is

$$\begin{aligned}&\delta z(t)\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)-\delta z(t_0)=\\&\quad\int_{t_0}^{t}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}\delta q_s+\frac{\partial L}{\partial \dot{q}_s}\delta\dot{q}_s\right)\mathrm{d}t\end{aligned} \tag{7}$$

From the initial condition $z(t)\big|_{t=t_0}=z_0$, and considering that the functional $z(t)$ reaches its extreme value at $t=t_1$, we obtain

$$\delta z(t_0)=\delta z(t_1)=0 \tag{8}$$

Since Eq.(7) is true for all $t\in[t_0,t_1]$, in particular, let $t=t_1$, we have

$$\int_{t_0}^{t}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}\delta q_s+\frac{\partial L}{\partial \dot{q}_s}\delta\dot{q}_s\right)\mathrm{d}t=0 \tag{9}$$

From Eq.(9), by using integration by parts and considering the endpoint conditions, we have

$$\int_{t_0}^{t_1}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_s}\right)\delta q_s\,\mathrm{d}t=0 \tag{10}$$

For the holonomic system, $\delta q_s(s=1,2,\cdots,n)$ are independent of each other, by the lemma of variation, we get

$$\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_s}\right)=0 \quad s=1,2,\cdots,n \tag{11}$$

Eq.(11) are called the Euler-Lagrange equations of Herglotz type for the holonomic nonconservative system.

If L does not contain $z(t)$, then Eq.(11) are reduced to Lagrange equations of the second kind for a holonomic conservative system.

1.3 Noether symmetry for the Lagrange system of Herglotz type

Noether symmetry of Herglotz type refers to the invariance of Hamilton-Herglotz action under the infinitesimal transformation of group, that is, $\Delta z(t_1)=0$ is always true for every infinitesimal transformation. The conserved quantity can be found by the Noether symmetry, and conversely, from a conserved quantity the corresponding Noether symmetry can be found.

For the holonomic nonconservative system (11), if the generating functions $\tau=\tau(t,q_k,\dot{q}_k,z)$ and $\xi_s=\xi_s(t,q_k,\dot{q}_k,z)$ of the infinitesimal transformation satisfy the following Noether identity

$$\frac{\partial L}{\partial t}\tau+\frac{\partial L}{\partial q_s}\xi_s+\frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s+\left(L-\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s\right)\dot{\tau}=0 \tag{12}$$

Then the Noether conserved quantity of Herglotz type exists, i.e.,

$$I_N=\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(L\tau+\frac{\partial L}{\partial \dot{q}_s}\bar{\xi}_s\right)=C \tag{13}$$

where $\bar{\xi}_s=\xi_s-\dot{q}_s\tau$.

Conversely, if there is a conserved quantity

$$I=I(t,q_k,\dot{q}_k,z)=C \tag{14}$$

Then the infinitesimal transformation determined by Eq.(15)

$$\bar{\xi}_s=\bar{h}_{sk}\frac{\partial I}{\partial \dot{q}_k}\exp\left(\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)$$
$$\xi_0=\frac{1}{L}\left[I\exp\left(\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)-\frac{\partial L}{\partial \dot{q}_s}\bar{\xi}_s\right] \tag{15}$$

is of Noether symmetry.

1.4 Generalization to nonholonomic dynamics

Assume that the motion of the nonconservative system is constrained by g first-order nonlinear nonholonomic constraints

$$\dot{q}_{\varepsilon+\beta}=\varphi_\beta(t,q_s,\dot{q}_\sigma)\quad \beta=1,2,\cdots,g;\varepsilon=n-g;\quad \sigma=1,2,\cdots,\varepsilon;s=1,2,\cdots,n \tag{16}$$

Then we have[11]

$$\int_{t_0}^{t_1}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial \widetilde{L}}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}+\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\frac{\partial L}{\partial z}+\frac{\partial \widetilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}-\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{\partial \varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}+\frac{\partial \varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial \varphi_\gamma}{\partial \dot{q}_\sigma}\right)\right]\delta q_\sigma \mathrm{d}t=0 \tag{17}$$

Considering the independence of δq_σ $(\sigma=1,2,\cdots,\varepsilon)$, we obtain the dynamical equations of Herglotz type for the nonholonomic nonconservative system, namely

$$\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial \widetilde{L}}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}+\frac{\partial L}{\partial z}\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}+\frac{\partial \widetilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}-\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{\partial \varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}+\frac{\partial \varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial \varphi_\gamma}{\partial \dot{q}_\sigma}\right)\right]=0\quad \sigma=1,2,\cdots,\varepsilon \tag{18}$$

If the generating function of space F_s and the generating function of time f satisfy the following restriction conditions

$$\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}(F_\sigma-\dot{q}_\sigma f)+\varphi_\beta f-F_{\varepsilon+\beta}=0\quad \beta=1,2,\cdots,g \tag{19}$$

and structural equation

$$\frac{\partial \widetilde{L}}{\partial q_s}F_s+\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\dot{F}_\sigma+\left(\widetilde{L}-\dot{q}_\sigma\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\right)\dot{f}+\frac{\partial \widetilde{L}}{\partial t}f-\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{\partial \varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}+\frac{\partial \varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial \varphi_\gamma}{\partial \dot{q}_\sigma}\right)(F_\sigma-\dot{q}_\sigma f)-\frac{\partial L}{\partial z}G+\dot{G}=0 \tag{20}$$

Then the system has the Noether conserved quantity of Herglotz type[11]

$$I=\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}F_\sigma+\left(\widetilde{L}-\dot{q}_\sigma\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\right)f+G\right]=C \tag{21}$$

1.5 Generalization to time-delay dynamics

Since time delay is a common phenomenon in nature and engineering practice, even a very simple problem may become very complicated once the influence of time delay is considered. In 1968, Hughes[12] studied the variational and optimal control problems with delayed argument and established the Euler-Lagrange equations with time delay. In 2012, Frederico and Torres[13] studied the Noether symmetry of variational and optimal control problems with time delay, and gave the Noether theorem of Lagrange system with time delay. In Ref.[14], we extended the results of Frederico and Torres in three aspects, that is, from Lagrange system to general nonconservative system; from point transformation group of generalized coordinates and time to infinitesimal transformation groups corresponding to generalized velocities, generalized coordinates and time; from Noether symmetry to Noether quasi-symmetry. Recently, Santos et al. studied the Herglotz variational problems with time delay and its Noether theorem[15], and extended the results to higher-order variational problems[16-17].

For the nonconservative dynamical system with time delay, the action functional satisfies the following first-order differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t}=L(t,q_s(t),\dot{q}_s(t),q_s(t-\tau),\dot{q}_s(t-\tau),z(t)) \tag{22}$$

where the time-delay quantity $\tau<t_1-t_0$ is a given positive real number, $q_s(t)=f_s(t)$ for $t\in[t_0-\tau,t_0]$, $f_s(t)$ are given piecewise smooth functions.

If the generating functions ξ_0 and ξ_s satisfy the following Noether identity

$$\begin{aligned}&\lambda(t+\tau)\left[\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\,\dot{q}_{s\tau}(t+\tau)\,\xi_0+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\,\ddot{q}_{s\tau}(t+\tau)\,\xi_0\right]=0,\ t\in[t_0-\tau,t_0)\\&\lambda(t)\left[\frac{\partial L}{\partial t}(t)\,\xi_0+\frac{\partial L}{\partial q_s}(t)\,\xi_s+\frac{\partial L}{\partial \dot{q}_s}(t)\,(\dot{\xi}_s-\dot{q}_s(t)\,\dot{\xi}_0)+L(t)\,\dot{\xi}_0\right]+\\&\lambda(t+\tau)\left[\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\,\xi_s+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\,(\dot{\xi}_s-\dot{q}_s(t)\,\dot{\xi}_0)\right]=0,\ t\in[t_0,t_1-\tau]\\&\lambda(t)\left[\frac{\partial L}{\partial t}(t)\,\xi_0+\frac{\partial L}{\partial q_s}(t)\,\xi_s+\frac{\partial L}{\partial \dot{q}_s}(t)\,(\dot{\xi}_s-\dot{q}_s(t)\,\dot{\xi}_0)+L(t)\,\dot{\xi}_0\right]=0,\ t\in(t_1-\tau,t_1]\end{aligned} \tag{23}$$

Then the following Noether conserved quantity of Herglotz type exists

$$\begin{aligned}&I_N=\lambda(t)\left[\frac{\partial L}{\partial \dot{q}_s}(t)\big(\xi_s-\dot{q}_s(t)\,\xi_0\big)+L(t)\,\xi_0\right]+\lambda(t+\tau)\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\big(\xi_s-\dot{q}_s(t)\,\xi_0\big),\ t\in[t_0,t_1-\tau]\\&I_N=\lambda(t)\left[\frac{\partial L}{\partial \dot{q}_s}(t)\big(\xi_s-\dot{q}_s(t)\,\xi_0\big)+L(t)\,\xi_0\right],\ t\in(t_1-\tau,t_1]\end{aligned} \tag{24}$$

where $\lambda(t)=\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)$.

1.6 Generalization to fractional dynamics

Fractional calculus, specifically, non-integral calculus, is an extension of integral calculus[18-19]. By using fractional calculus to model dynamics, the dynamical behavior and physical essence of complex systems can be described more accurately. Riewe[20] was the first scientist to introduce fractional calculus into dynamic modeling of nonconservative systems. Agrawal[21] proposed fractional variational problem and derived fractional Euler-Lagrange equations, whose form is similar to that obtained by classical integer order variational problem. In Refs.[22-28], the authors studied fractional variational problems and fractional Noether symmetry and conserved quantities under the framework of Lagrange.

In Ref.[29], two methods dealing with nonconservative systems, namely, Herglotz's generalized variational principle and fractional calculus method, were combined to propose the Herglotz variational problems with Caputo fractional derivatives in functional differential equations, as well as the fractional variational problems of Herglotz type with several independent variables. The fractional Euler-Lagrange equations of Herglotz type and corresponding transversality conditions were established, the fractional Noether theorem was proved, and the fractional conserved quantity of Frederico-Torres type was given. In Ref.[30], the fractional Herglotz variational principle with generalized Caputo derivatives was studied, and its application is discussed by taking the damped harmonic oscillator whose mass and elastic stiffness change with time as an example. The fractional Herglotz variational problems of variable order are considered in Ref. [31]. There are also researches on the fractional variational problem of Herglotz type under the framework of Lagrange, for example, see Refs. [32-34].

1.7 Generalization to time-scale dynamics

Time-scale analysis theory is a mathematical theory proposed by German scholar Hilger in 1988[35]. The dynamical equations on time scales integrate the differential equations dealing with continuous system and the difference equations dealing with discrete system, which can not only reveal the similarities and differences between continuous and discrete systems, but also describe the physical essence of continuous and discrete systems and other complex dynamical systems more clearly and accurately[36-37]. In 2004, Bohner[38] first investigated the variational problem on time scales and established the corresponding Euler-Lagrange equations and transversality conditons. In 2008, Bartosiewicz et al.[39] first proved the Noether theorem of Lagrange system on time scales. Recently, we have studied the dynamics of nonholonomic Chaplygin systems on time scales, the method of reduction for nonconservative Lagrange systems, and the Noether theorem on time scales[40-42].

In Ref.[43], we extended Herglotz's generalized variational principle to time-scale dynamical systems and established the Euler-Lagrange equations of Herglotz type for Lagrange system on time scales. We studied the Noether symmetry of Herglotz variational problem on time scales, derived the Noether identity, obtained the Noether conserved quantity, and proved the Noether theorem of Herglotz type for Lagrange systems on time scales. If the time scale $\boldsymbol{T}$ is taken as a set of real numbers $\boldsymbol{R}$, the results are degraded to Herglotz's generalized variational principle, dynamical equations, and Noether theorem of continuous nonconservative Lagrange systems. If the time scale $\boldsymbol{T}$ is taken as a set of integers $\boldsymbol{Z}$, the results are reduced to Herglotz's generalized variational principle, dynamical equations and Noether theorem of discrete nonconservative Lagrange systems. Obviously, the Herglotz variational problem and its symmetry on time scales is an open issue.

1.8 Other generalization

Santos et al.[44-45] studied the higher-order Herglotz variational problems with the Lagrangian containing higher-order derivatives, and gave the generalized Euler-Lagrange equations and the transversality conditions, as well as Noether theorem. Ref. [46] studied Herglotz generalized variational problems from the perspective of optimal control. In Refs.[47-48], the authors extended the variational problem of Herglotz type to the more general context of the Euclidean sphere following variational and optimal control approaches. Lazo et al.[49-50] applied Herglotz's generalized variational principle to several nonconservative classical and quantum systems, constructed Lagrangians in the sense of Herglotz, and explained the physical meaning of Lagrangians.

1.9 Problems to be further studied in Lagrangian mechanics of Herglotz type

For the Lagrangian mechanics of Herglotz type, we propose some problems to be further studied as follows:

Problem 1.1 How to construct the Lagrangians in the sense of Herglotz for general nonconservative dynamical systems? What is the physical meaning of Lagrangians of Herglotz type? What is the relationship and difference between Lagrangians of Herglotz type and classical Lagrangians, i.e., kinetic energy minus potential energy?

Problem 1.2 How to establish the formulation of Lie symmetry and conserved quantity theory for Lagrangian mechanics of Herglotz type and how to solve it? What is the relationship between Lie symmetry of Herglotz type and Noether symmetry?

Problem 1.3 How to establish the formulation of Mei symmetry and conserved quantity theory for Lagrangian mechanics of Herglotz type and how to solve it? What is the relationship between Mei symmetry of Herglotz type and Noether symmetry?

Problem 1.4 How to establish Herglotz's generalized variational principle for multivariable Lagrange system on time scales and how to find its formulation and solution of symmetry and conserved quantity theory?

Problem 1.5 Since Herglotz's generalized variational principle provides a variational description of nonconservative systems, can and how to construct structure preserving algorithms for nonconservative Lagrange systems?

Problem 1.6 How to establish the geometric theory of Lagrangian mechanics of Herglotz type?

2 Hamiltonian Mechanics of Herglotz Type

Since the phase space of a dynamical system has natural symplectic structure, it is easier to describe mathematically than Lagrangian mechanics. For some dynamical systems, it is not obviously symmetric in configuration space, but has some symmetric property in phase space, and the corresponding conserved quantity can be found by using the canonical form of Noether theorem. This section introduces Herglotz's generalized variational principle for nonconservative systems in phase space and some generalizations of it.

2.1 Herglotz's generalized variational principle

The variational problem of Herglotz type for a nonconservative system in phase space can be formulated as follows[51]:

Determine the trajectories $q_s(t)$ and $p_s(t)$ satisfying the given endpoint conditions $q_s(t)\big|_{t=t_0}=q_{s0}$, $q_s(t)\big|_{t=t_1}=q_{s1}$ $(s=1,2,\cdots,n)$ that extremize the value $z(t_1)\rightarrow$ extr, where the functional $z(t)$ is defined by the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t}=p_s(t)\dot{q}_s(t)-H(t,q_s(t),p_s(t),z(t)) \qquad t\in[t_0,t_1] \tag{25}$$

and subjected to the initial condition $z(t)\big|_{t=t_0}=z_0$. $H(t,q_s(t),p_s(t),z(t))$ is called the Hamiltonian in the sense of Herglotz, $q_s(t)$ are the generalized coordinates, and $p_s(t)$ are the generalized momentums.

The aforementioned variational problem can be referred to as Herglotz's generalized variational principle for nonconservative system in phase space.

If H does not contain $z(t)$, then the functional $z(t)$ becomes the classical Hamilton action in phase space, i.e.

$$z=\int_{t_0}^{t_1}[p_s(t)\dot{q}_s(t)-H(t,q_s(t),p_s(t))]\mathrm{d}t \tag{26}$$

And Herglotz's principle is reduced to the classical Hamilton principle for holonomic conservative system in phase space $\delta z=0$.

2.2 Hamilton canonical equations of Herglotz type

According to Herglotz's generalized variational principle for nonconservative systems in phase space, we can easily deduce

$$\exp\left(\int_{t_0}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left[\left(-\dot{p}_s-\frac{\partial H}{\partial q_s}-p_s\frac{\partial H}{\partial z}\right)\delta q_s+\left(\dot{q}_s-\frac{\partial H}{\partial p_s}\right)\delta p_s\right]\mathrm{d}t=0 \tag{27}$$

Since δq_s and δp_s are independent of each other, by using the lemma of variation, we get

$$\begin{gathered}\exp\left(\int_{t_0}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(\dot{q}_s-\frac{\partial H}{\partial p_s}\right)=0\\ \exp\left(\int_{t_0}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\left(-\dot{p}_s-\frac{\partial H}{\partial q_s}-p_s\frac{\partial H}{\partial z}\right)=0\\ s=1,2,\cdots,n\end{gathered} \tag{28}$$

Eq.(28) is called the Hamilton canonical equations of Herglotz type for the holonomic nonconservative system in phase space.

If H does not contain $z(t)$, then Eq.(24) is reduced to the classical Hamilton canonical equations for holonomic conservative systems.

2.3 Noether symmetry for the Hamilton system of Herglotz type

For the holonomic nonconservative system (28) in phase space, if the generating functions $\tau=\tau(t,q_k,p_k,z)$, $\xi_s=\xi_s(t,q_k,p_k,z)$ and $\eta_s=\eta_s(t,q_k,p_k,z)$ of the infinitesimal transformation satisfy the following Noether identity

$$\dot{q}_s\eta_s+p_s\dot{\xi}_s-H\dot{\tau}-\frac{\partial H}{\partial t}\tau-\frac{\partial H}{\partial q_s}\xi_s-\frac{\partial H}{\partial p_s}\eta_s=0 \tag{29}$$

Then the Noether conserved quantity of Herglotz type exists, i.e.

$$I_N=\left(p_s\xi_s-H\tau\right)\exp\left(\int_{t_0}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)=C \tag{30}$$

Conversely, if there is a conserved quantity

$$I=I(t,q_s,p_s,z)=C \tag{31}$$

Then the infinitesimal transformation is determined by the following formulas

$$\xi_s=\frac{\partial H}{\partial p_s}\tau+\frac{\partial I}{\partial p_s}\exp\left(-\int_{t_0}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\quad s=1,2,\cdots,n$$
$$\tau=\frac{1}{H}\left[p_s\xi_s-I\exp\left(-\int_{t_0}^{t}\frac{\partial H}{\partial z}\mathrm{d}\theta\right)\right] \tag{32}$$

is of Noether symmetry.

2.4 Generalization to time-delay dynamics

The study of dynamics in phase space with time delay is a new subject. In Ref.[52], we studied the variational principle of Hamilton system with time delay based on the classical variational principle, and derived the Hamilton canonical equations with time delay. The definition and criterion of Noether symmetric transformations and quasi-symmetric transformations for Hamilton system with time delay were studied, and the Noether theorem of the system was proved. Recently, based on Herglotz's generalized variational principle, we studied the dynamics of nonconservative systems with time delays in phase space[53], gave a variational description of the system, derived the Hamilton canonical equations of Herglotz type for holonomic nonconservative systems with time delay, established the definition and criterion of the Noether symmetry, and proved the Noether theorem of the system.

If the delay does not exist, the results of the time-delay dynamics are reduced to those of Section 2.1, 2.2, and 2.3. If H does not contain $z(t)$, it is reduced to Hamilton canonical equations with time delay and Noether's theorem based on the classical Hamilton variational principle[52].

2.5 Generalization to fractional dynamics

Some papers on Hamiltonian mechanics under fractional order model have been published, such as Refs.[54-57]. Muslih and Baleanu[54] studied the fractional Hamiltonian mechanics with singular Lagrangian by means of Riemann-Liouville derivatives. Li and Luo[55] studied the generalized Hamiltonian mechanics. In Refs.[56-57], the fractional Hamiltonian mechanics and its canonical transformations and fractional Noether theorem are studied. Recently, the fractional Herglotz variational problem in phase space was studied in Ref.[58]. Based on Caputo fractional derivative, the fractional Herglotz's generalized variational principle in phase space was established, and the fractional Hamilton canonical equations of Herglotz type were derived. The invariance of Hamilton-Herglotz action in phase space under infinitesimal transformation was studied, and the fractional Noether theorem of Herglotz type in phase space was proved.

2.6 Generalization to time-scale dynamics

Recently, we have studied the symmetry and conserved quantity of Hamilton system on time scales and its Hamilton-Jacobi method[59-61]. In Ref.[62], we proposed Herglotz's generalized variational principle for nonconservative Hamilton system on time scales, and derived the Hamilton canonical equations of Herglotz type on time scales, and proved the Noether theorem of the system. In general, Hamiltonian dynamics and its generalization to Herglotz's generalized variational principle on time scales are still in the initial stage.

2.7 Problems to be further studied in Hamiltonian machanics of Herglotz type

For the Hamiltonian mechanics of Herglotz type, we propose some problems to be further studied as follows:

Problem 2.1 What is the physical meaning of Hamiltonian of Herglotz type? How does it relate to classical Hamiltonian? How to construct the Hamiltonian of Herglotz type for a real dynamical system?

Problem 2.2 How to establish the theory of Lie symmetry and Mei symmetry for Hamiltonian mechanics of Herglotz type? What is the relationship between the symmetry and conserved quantity theory for Hamiltonian mechanics of Herglotz type and those of Lagrangian mechanics?

Problem 2.3 Can and how to construct the Hamilton-Jacobi theory for nonconservative dynamics based on Herglotz's generalized variational principle?

Problem 2.4 How to construct the geometric theory for Hamiltonian mechanics of Herglotz type?

Problem 2.5 Based on Herglotz's generalized variational principle, can and how to extend the theory of Hamilton symplectic geometric algorithm to nonconservative system and construct structure preserving algorithms for nonconservative Hamilton systems?

Problem 2.6 How to construct Herglotz's generalized variational principle for multivariable dynamical system in phase space? How to study multivariable Hamiltonian mechanics of Herglotz type?

3 Birkhoffian Mechanics of Herglotz Type

Birkhoffian mechanics is a natural development of Hamiltonian mechanics and a new stage in the development of analytical mechanics. It is widely used in the fields of mechanics, physics and engineering[63]. Birkhoff proposed a new class of integral variational principles and a new class of differential equations of motion in his works "Dynamical systems"[64]. Santilli, in his monograph[65], proposed the term "Birkhoffian mechanics" on the basis of the differential equations and studied Birkhoff's equation, the transformation theory of Birkhoff's equation and the generalization of Galilei's relativity. At the same time, he made a good summary of the origin of Birkhoff's equations and subsequent research. Mei called this principle Pfaff-Birkhoff principle in his monograph[66], and studied the Birkhoffian theory of holonomic and non-holonomic mechanics, integration theory of Birkhoff system, dynamical inverse problem, motion stability, geometric method and global analysis and so on, which enriched and developed Birkhoffian mechanics. This section introduces Herglotz's generalized variational principle for Birkhoff systems and its generalization.

3.1 Herglotz's generalized variational principle

The variational problem of Herglotz type for a Birkhoff system can be formulated as follows[67]

Determine the trajectories $a^\mu(t)$ satisfying given endpoint conditions $a^\mu(t)\big|_{t=t_0}=a_0^\mu$, $a^\mu(t)\big|_{t=t_1}=a_1^\mu$ $(\mu=1,2,\cdots,2n)$ that extremize the value $z(t_1)\rightarrow$ extr, where the functional $z(t)$ is defined by the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t}=R_\mu(t,a^\nu(t),z(t))\dot{a}^\mu(t)-B(t,a^\nu(t),z(t))\qquad t\in[t_0,t_1] \tag{33}$$

and subjected to the initial condition $z(t)\big|_{t=t_0}=z_0$. Here, $B(t,a^\nu(t),z(t))$ and $R_\mu(t,a^\nu(t),z(t))$ can be called the Birkhoffian and Birkhoff's functions in the sense of Herglotz, $a^\mu(t)$ are the Birkhoff's variables, and a_0^μ, a_1^μ and z_0 the fixed real constants.

Functional $z(t)$ is called Pfaff-Herglotz action. The variational problem mentioned above can be referred to as the Herglotz's generalized variational principle for Birkhoffian mechanics.

If B and R_μ do not contain z, then the Pfaff-Herglotz action becomes the classical Pfaff action, i.e.

$$z=\int_{t_0}^{t_1}[R_\mu(t,a^\nu(t))\dot{a}^\mu(t)-B(t,a^\nu(t))]\,\mathrm{d}t \tag{34}$$

And the principle is reduced to the classical Pfaff-Birkhoff principle.

3.2 Birkhoff's equations of Herglotz type

According to Herglotz's generalized variational principle for Birkhoff systems, we can easily derive

$$\int_{t_0}^{t_1}\exp\left[-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right]\cdot \left\{\left(\frac{\partial R_\upsilon}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\upsilon}\right)\dot{a}^\upsilon-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}+\left(R_\mu\frac{\partial R_\upsilon}{\partial z}-\frac{\partial R_\mu}{\partial z}R_\upsilon\right)\dot{a}^\upsilon+\frac{\partial R_\mu}{\partial z}B-R_\mu\frac{\partial B}{\partial z}\right\}\delta a^\upsilon\mathrm{d}t=0 \tag{35}$$

Since δa^μ ($\mu=1,2,\cdots,2n$) are independent of each other, by using the lemma of variation, we obtain

$$\exp\left[-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right]\left\{\left(\frac{\partial R_\upsilon}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\upsilon}\right)\dot{a}^\upsilon-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}+\left(R_\mu\frac{\partial R_\upsilon}{\partial z}-\frac{\partial R_\mu}{\partial z}R_\upsilon\right)\dot{a}^\upsilon+\frac{\partial R_\mu}{\partial z}B-R_\mu\frac{\partial B}{\partial z}\right\}=0$$
$$\mu=1,2,\cdots,2n \tag{36}$$

Eq. (36) is called the Birkhoff's equations of Herglotz type for the Birkhoff system. If B and R_μ do not contain z, then Eq. (36) is reduced to the classical Birkhoff's equation.

3.3 Noether symmetry for the Birkhoff system of Herglotz type

For the Birkhoff system (36) of Herglotz type, if the generating functions $\tau=\tau(t,a^\upsilon,z)$ and $\xi_\mu=\xi_\mu(t,a^\upsilon,z)$ of the infinitesimal transformation satisfy the following Noether identity

$$\left(\frac{\partial R_\mu}{\partial t}\tau+\frac{\partial R_\mu}{\partial a^\upsilon}\xi_\upsilon\right)\dot{a}^\mu+R_\upsilon\dot{\xi}_\upsilon-B\dot{\tau}-\frac{\partial B}{\partial a^\upsilon}\xi_\upsilon-\frac{\partial B}{\partial t}\tau=0 \tag{37}$$

Then the Noether conserved quantity of Herglotz type exists, i.e.

$$I_N=\left(R_\mu\xi_\mu-B\tau\right)\exp\left(-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right)=C \tag{38}$$

Conversely, if there is a conserved quantity

$$I=I(t,a^\upsilon,z)=C \tag{39}$$

Then the infinitesimal transformation determined by the following formulas

$$\frac{\partial I}{\partial a^\upsilon}+\frac{\partial I}{\partial z}R_\upsilon+\exp\left[-\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right]\cdot\left\{\left(\frac{\partial R_\upsilon}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\upsilon}\right)\xi_\mu+\left(R_\mu\frac{\partial R_\upsilon}{\partial z}-\frac{\partial R_\mu}{\partial z}R_\upsilon\right)\xi_\mu+\left(\frac{\partial B}{\partial a^\upsilon}+\frac{\partial R_\upsilon}{\partial t}-\frac{\partial R_\upsilon}{\partial z}B+R_\upsilon\frac{\partial B}{\partial z}\right)\tau\right\}=0$$
$$\upsilon=1,2,\cdots,2n$$
$$\tau=\frac{1}{B}\left\{R_\mu\xi_\mu-I\exp\left(\int_{t_0}^{t}\left(\frac{\partial R_\mu}{\partial z}\dot{a}^\mu-\frac{\partial B}{\partial z}\right)\mathrm{d}\theta\right)\right\} \tag{40}$$

is of Noether symmetry.

3.4 Generalization to constrained Birkhoff system

The differential variational principle Pfaff-Birkhoff-d'Alembert principle of Herglotz type was given in Ref. [68]. Based on this principle, Birkhoff's equations of Herglotz type for the constrained Birkhoff system were established by using Lagrange multiplier method. According to the relation between the isochronal variation and the nonisochronal variation, the condition of the invariance for the Pfaff-Birkhoff-d'Alembert principle of Herglotz type was given, and the Noether conserved quantity for the Birkhoff system and the constrained Birkhoff system of Herglotz type was obtained. The conserved quantity obtained is more general than that in Ref.[67].

3.5 Generalization of time-delay dynamics

Some papers on the dynamics of Birkhoff systems with time delays have been published[69-71]. In Ref. [69], we studied the Noether symmetry and conserved quantity of Birkhoff systems with time delay, established the Pfaff-Birkhoff principle and Birkhoff's equations with time delay, and proved the Noether theorem of Birkhoff systems with time delay. In Refs.[70-71], we further studied Noether symmetry and conserved quantity for fractional Birkhoff systems with time delay and generalized Birkhoff systems with time delay. Recently, we have combined the Birkhoffian mechanics with Herglotz's generalized variational principle, established Birkhoff's equations of Herglotz type with time delay, and proved the Noether theorem for the Birkhoff system of Herglotz type with time delay[72].

3.6 Generalization to fractional dynamics

We also applied fractional calculus to the investigation of Birkhoffian dynamics and its symmetry theory[73-76]. In Ref.[73], we proposed the fractional Pfaff-Birkhoff principle based on the Riemann-Liouville fractional derivatives, and derived the fractional Birkhoff's equations. The Noether symmetry was studied and the fractional Noether conserved quantity was given based on the Frederico-Torres definition of fractional conserved quantity. Different from Ref.[73], in Ref.[74] we used the classical definition of conserved quantity to give the Noether conserved quantity of fractional Birkhoff systems and prove the Noether theorem. The Noether theorem for quasi-fractional Birkhoff systems was given in Ref.[75]. The Herglotz variation problem of fractional Birkhoff system was proposed and studied in Ref.[76]. The fractional Birkhoff's equations of Herglotz type were established under the derivatives of Riemann-Liouville, Caputo, and Riesz, respectively. Noether symmetry of the fractional Birkhoff systems was studied, and the Noether theorem of Herglotz type for the fractional Birkhoff system was derived.

3.7 Generalization to time-scale dynamics

In 2015, we established the Pfaff-Birkhoff principle with delta derivatives on time scales, derived the dynamical equations of Birkhoff system on time scales, and studied the Noether symmetry and conserved quantity of Birkhoff system on time scales, and proved the Noether theorem[77]. In Ref.[78], we proved the Pfaf-Birkhoff principle, Birkhoff's equations and Noether theorem of Birkhoff system under nabla derivatives by using dual principle. In Ref.[79], we studied the Lie symmetry and conserved quantity of Birkhoff system on time scales, and gave the adiabatic invariants led by Lie symmetry under small perturbations. Recently, we have studied Herglotz's generalized variational principle for Birkhoff system on time scales and derived Birkhoff's equations of Herglotz type with delta derivatives. Based on the invariance of the Pfaff-Birkhoff action on time scales under infinitesimal transformations, the Noether identity of Herglotz type for the Birkhoff system with delta derivatives was established, and the Noether theorem of Herglotz type for the Birkhoff system with time scales was proved. So far, studies on Birkhoffian mechanics of Herglotz type on time scales have not been reported.

3.8 Problems to be further studied in Birkhoffian mechanics of Herglotz type

For the Birkhoffian mechanics of Herglotz type, we propose some problems to be further studied as follows:

Problem 3.1 How to construct the Birkhoffian and Birkhoff's functions of Herglotz type? How do they relate to the classical Birkhoffian and Birkhoff's functions?

Problem 3.2 How to establish the Lie symmetry, the Mei symmetry and the corresponding conserved quantity theory for the Birkhoff system of Herglotz type?

Problem 3.3 How to extend the integral theory of the classical Birkhoff system, for example, generalized canonical transformation theory, Poisson theory, generalized Hamilton-Jacobi theory, etc. to the Birkhoff system of Herglotz type?

Problem 3.4 How to construct the geometric theory for Birkhoffian mechanics of Herglotz type?

Problem 3.5 How to construct structure preserving algorithms for Birkhoffian mechanics of Herglotz type?

Problem 3.6 How to construct Herglotz's generalized variational principle for multivariable Birkhoff systems? How to study multivariable Birkhoffian mechanics of Herglotz type?

4 Conclusions

Herglotz's generalized variational principle provides a method for dynamic modeling of nonconservative or dissipative systems. The variational description of nonconservative systems can be realized by this method. We gave a brief review of Herglotz's generalized variational principle and the research status of Noether symmetry and conserved quantities from the perspectives of Lagrangian mechanics, Hamiltonian mechanics and Birkhoffian mechanics, as well as the generalization to time delay dynamics, fractional dynamics and time scale dynamics. Some problems to be further studied were put forward, which can be helpful to the further development of Herglotz's generalized variational principle and its application.

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非保守动力学的Herglotz广义变分原理的研究进展

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摘要:综述非保守动力学系统的Herglotz广义变分原理及其对称性与守恒量研究的最新进展。以Lagrange力学、Hamilton力学和Birkhoff力学作为研究框架,介绍其Herglotz广义变分原理、Herglotz型动力学方程、Noether对称性与守恒量,以及对时滞动力学、分数阶动力学、时间尺度动力学的推广,并提出若干问题作为未来研究的建议。

关键词:非保守动力学;Herglotz广义变分原理;Lagrange力学;Hamilton力学;Birkhoff力学