Three-Dimensional Thermal-Stress Analysis of Semi-infinite Transversely Isotropic Composites

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Abstract: By making use of the direct integration method, an exact analysis of the general three-dimensional thermoelasticity problem is performed for the case of a transversely isotropic homogeneous half-space subject to local thermal and force loadings. The material plane of isotropy is assumed to be parallel to the limiting surface of the half-space. By reducing the original thermoelasticity equations to the governing ones for individual stress-tensor components, the effect of material anisotropy in the stress field is analyzed with regard to the feasibility requirement, i.e., the finiteness of the stress field at a distance from the disturbed area. As a result, the solution is constructed in the form of explicit analytical dependencies on the force and thermal loadings for various kinds of transversely isotropic materials and agrees with the basic principles of the continua mechanics. The solution can be efficiently used as a benchmark one for the direct computation of temperature and thermal stresses in transversely isotropic semi-infinite domains, as well as for the verification of solutions constructed by different means.

Key words: three-dimensional problem; analytical solution; transversely isotropic composites; semi-infinite model; force and thermal loadings; finite stress distributions

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0 Introduction

Transversely isotropic materials can be regarded as the simplest example of materials exhibiting spatial anisotropy. Due to the specific micro-structure (i.e., the structure of molecular lattice or specific features of the material composition), the elastic and thermo-physical properties of a macro-volume remain the same within a certain plane, which is known as the plane of isotropy[1-2], but are different from the ones in the direction that is perpendicular to the plane mentioned. The symmetry of such kind is typical for a number of natural and composite materials, e.g., the ones with lattices of hexagonal syngony[3], as well as angle-ply laminates[4-5] or fiber composites with hexagonal packing[6], etc., which can be regarded as homogeneous transversely isotropic solids after utilization of certain homogenization techniques[7-10].

Despite the apparent simplicity of the transversely isotropic material in comparison with the materials of rather general anisotropy, the dissimilarity of effective properties presents a certain challenge for the analysis of the relevant three-dimensional problems of mechanics. Being involved in the constitutive equations of the corresponding mathematical model, the elastic and thermoelastic moduli of a transversely isotropic material are thereby affecting

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the coefficients of the governing equations of thermoelasticity theory[11]. Thus, the form of a solution to the corresponding governing equations strongly depends on the interrelations between the material moduli. This presents a challenge for the general analysis of thermal stresses in transversely isotropic solids due to the fact that a solution is to cover interrelations between material moduli of any kind.

This problem becomes even more involved when analyzing local effects of force or thermal loadings. One of the basic models frequently used for such analysis deals with an elastic half-space whose surface is acted upon by locally distributed impacts[12-13]. Besides the satisfaction of the boundary conditions, the solutions of such elasticity and thermoelasticity problems are to exhibit asymptotic behavior that is vanishing at a distance from the loaded zones, which meets the feasible requirements of the Saint-Venant's principle[14]. Because of uncertain interrelation between the coefficients of the governing equations (which, in turn, depend on the material moduli) for any given kind of a transversely isotropic material, ensuring the required asymptotic behavior of the solution remains an important and yet unanswered challenge (e.g., the reviews[15-18]) especially for analytical methods based on the application of potential functions of higher differential rate.

An efficient technique for the analysis of anisotropic and inhomogeneous solids has been developed on the basis of the direct integration method[19-20]. This technique has also been extended onto the cases of three-dimensional problems for transversely isotropic solids[15-17]. It implies a reduction of the original thermoelasticity equations to a set of governing equations for individual stress-tensor components with accompanying local and integral boundary conditions. The fact of getting an individual equation for a stress-tensor component can be effectively used for the more accurate evaluation of the stress asymptotic for transversely isotropic semi-infinite solids.

This paper presents an attempt towards the construction of analytical solutions to a three-dimensional thermoelasticity problem for a transversely isotropic half-space, which meets the original equations along with the given boundary conditions and is vanishing at a distance from the loaded zones of the limiting surface or inner heat-sources.

1 Formulation of the Problem

Consider a three-dimensional problem of the thermoelasticity theory for a transversely isotropic half-space $(x, y, z) \in \mathbf{R}^2 \times \mathbf{R}_+$ in the dimensionless Cartesian coordinate system. Let the plane of isotropy be parallel to the limiting surface $z=0$. Within the framework of the quasi-static formulation in the absence of body forces, the problem is governed[1, 2, 14] by the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}=0 \qquad \begin{matrix} & y & \\ \nearrow & & \searrow \\ x & \leftarrow & z \end{matrix} \tag{1}$$

the strain-compatibility equations

$$\begin{gathered} \frac{\partial^2 \varepsilon_{xx}}{\partial y^2}+\frac{\partial^2 \varepsilon_{yy}}{\partial x^2}=\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \\ 2\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z}=\frac{\partial}{\partial x}\left(\frac{\partial \varepsilon_{xy}}{\partial z}-\frac{\partial \varepsilon_{yz}}{\partial x}+\frac{\partial \varepsilon_{xz}}{\partial y}\right) \qquad \begin{matrix} & y & \\ \nearrow & & \searrow \\ x & \leftarrow & z \end{matrix} \end{gathered} \tag{2}$$

and the constitutive ones

$$\begin{gathered} \varepsilon_{xx}=\frac{1}{E}(\sigma_{xx}-\nu\sigma_{yy})-\frac{\nu_z}{E_z}\sigma_{zz}+\alpha T \quad x \rightleftarrows y \\ \varepsilon_{zz}=\frac{1}{E_z}\sigma_{zz}-\frac{\nu_z}{E_z}(\sigma_{xx}+\sigma_{yy})+\alpha_z T \\ \varepsilon_{jz}=\frac{1}{G_z}\sigma_{jz}, \varepsilon_{xy}=\frac{1}{G}\sigma_{xy} \quad j=\{x, y\} \end{gathered} \tag{3}$$

where $\sigma_{\xi\eta}=\sigma_{\eta\xi}$, $\varepsilon_{\xi\eta}=\varepsilon_{\eta\xi}$ are the stress- and strain-tensor components, $\xi, \eta=\{x, y, z\}$; E, E_z and G, G_z are the in- and out-of-plane (with respect to the plane of isotropy) Young and shear moduli, respectively; ν, ν_z and α, α_z are the transversely isotropic Poisson ratios and the linear thermal expansion coefficients; symbols $\begin{matrix} & y & \\ \nearrow & & \searrow \\ x & \leftarrow & z \end{matrix}$ and $x \rightleftarrows y$ imply obtaining two and one more equation from the one they follow by the cyclic and mutual, respectively, permutations of indices and variables.

The stationary temperature field $T(x, y, z)$ can be determined from the following heat-transfer equation[21]

$$c\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+c_z\frac{\partial^2 T}{\partial z^2}=-Q \tag{4}$$

under the general boundary condition

$$\left(aT+b\frac{\partial T}{\partial z}\right)\Big|_{z=0}=T_0 \tag{5}$$

where c and c_z are the heat conductivity coefficients within the plane of isotropy and transversely; $Q(x,y,z)$ and $T_0(x,y)$ are the given densities of internal heat sources and thermal boundary function, both vanishing at $x^2+y^2+z^2\to+\infty$ and $x^2+y^2\to+\infty$, respectively. Constant parameters a and b indicate the type of boundary condition (5): If $a\neq0$ and $b=0$, Eq.(5) is the Dirichlet condition imposing the temperature on the boundary; if $a=0$ and $b\neq0$, Eq.(5) is the Neumann condition imposing the heat flux through the boundary; if $a\neq0$ and $b\neq0$, Eq.(5) is the third-kind boundary condition covering, for example, the heat-exchange through the boundary[22].

Our intent is to construct an analytical solution to the formulated thermoelasticity Eqs.(1—3) under the temperature field determined from the heat-conduction problem Eqs.(4, 5) and the force loadings

$$\sigma_{zz}\big|_{z=0}=-p,\ \sigma_{\xi z}\big|_{z=0}=q_\xi\quad \xi=\{x,y\} \tag{6}$$

imposed on the boundary $z=0$, where $p(x,y)$ and $q_\xi(x,y)$ are given functions vanishing at $x^2+y^2\to+\infty$. Making use of equilibrium equation (1) and boundary conditions (6) for the tangential stress yields the condition

$$\frac{\partial\sigma_{zz}}{\partial z}\Big|_{z=0}=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}\right) \tag{7}$$

for the partial derivative by z of the normal stress. For the complete determination of the stress field, the strain-compatibility condition in the following integral form[11] is to be used.

$$\int_{-\infty}^{x}\left(\varepsilon_{xz}(\xi,y,0)-\int_{-\infty}^{\xi}\frac{\partial\varepsilon_{xx}(\xi_1,y,0)}{\partial z}\mathrm{d}\xi_1\right)\mathrm{d}\xi=\int_{-\infty}^{y}\left(\varepsilon_{yz}(x,\eta,0)-\int_{-\infty}^{\eta}\frac{\partial\varepsilon_{yy}(x,\eta_1,0)}{\partial z}\mathrm{d}\eta_1\right)\mathrm{d}\eta \tag{8}$$

2 Solution Method

To separate variables in the foregoing thermoelasticity and heat conduction equations and boundary conditions, we employ the Fourier double-integral transform[23]

$$\bar f(z)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y,z)\exp(-\mathrm{i}(xs_x+ys_y))\mathrm{d}x\mathrm{d}y \tag{9}$$

where s_x and s_y are the transform parameters with respect to x, y and i is the imaginary unit.

Making use of transform (9) allows for solving the problems (4) and (5) in the Fourier mapping domain in the form as follows

$$\bar T(z)=\frac{\mathrm{e}^{-c_0|s|z}}{a^-}\bar T_0+\frac{1}{2c_0|s|a^-}\int_0^{+\infty}\bar q(\zeta)(a^-\mathrm{e}^{-c_0|s(z-\zeta)|}-a^+\mathrm{e}^{-c_0|s|(z+\zeta)})\mathrm{d}\zeta \tag{10}$$

where $a^\pm=a\pm c_0|s|b$, $c_0=\sqrt{c/c_z}>0$, $\bar q=\bar Q/c_z$. Also, here and in what follows, $|s|$ indicates the absolute value of s and $s^2=s_x^2+s_y^2$.

By implementing the technique[15, 24], we can reduce the formulated thermoelasticity problem (1—3) to the following system of governing equations in terms of stresses

$$\Delta^+\Delta^-\sigma_{zz}-\mu^+\mu^-\Delta_{xy}\Delta_1\sigma_{zz}=\mu^+E\Delta_{xy}\times\left((\alpha+(\alpha+\alpha_z)\mu^-)\Delta_{xy}T+\alpha(1+\nu)\mu^-\frac{\partial^2T}{\partial z^2}\right) \tag{11}$$

$$\Delta^+\sigma_{zz}=\mu^+\Delta_{xy}(\sigma+\alpha ET) \tag{12}$$

$$\Delta_{xy}\sigma_{yy}+\mu_1\frac{\partial^2\sigma_{yy}}{\partial z^2}=\left(\frac{\mu_2}{\mu^+}-1\right)\Delta\sigma_{zz}+(1-\mu_2)\frac{\partial^2\sigma}{\partial x^2}+(\mu_1-\mu_2)\frac{\partial^2\sigma}{\partial z^2}+\frac{\partial^2}{\partial y^2}\left(\frac{\mu_2\sigma}{\mu^-}+(1-\mu_3)\sigma_{zz}\right)-2G_z\left(\alpha\Delta T+\alpha_z\frac{\partial^2T}{\partial y^2}\right)\qquad x\rightleftarrows y \tag{13}$$

$$2\frac{\partial^2\sigma_{xy}}{\partial x\partial y}=\frac{\partial^2\sigma_{zz}}{\partial z^2}-\frac{\partial^2\sigma_{yy}}{\partial y^2}-\frac{\partial^2\sigma_{xx}}{\partial x^2}\qquad \begin{matrix} & y & \\ \nearrow & & \searrow \\ x & \leftarrow & z\end{matrix} \tag{14}$$

where

$$\Delta_{xy}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\Delta=\Delta_{xy}+\frac{\partial^2}{\partial z^2}$$

$$\Delta^\pm=\Delta_{xy}\pm(1\pm\nu)\mu^\pm\frac{\partial^2}{\partial z^2},\mu^\pm=\frac{E_z}{\nu_zE\pm E_z}$$

$$\mu_1=\frac{G_z}{G},\Delta_1=(\mu_4-1)\Delta_{xy}+2\frac{\mu^+-\mu_2}{\mu_2\mu^+}\frac{\partial^2}{\partial z^2}$$

$$\mu_2=2\frac{G_z}{E},\mu_3=2\frac{1+\nu_z}{E_z}G_z,\mu_4=\frac{E}{E_z}$$

and

$$\sigma=\sigma_{xx}+\sigma_{yy}+\sigma_{zz} \tag{15}$$

Having applied transform (9) to Eq.(11) and conditions (6) and (7), we obtain the following boundary-value problem

$$\frac{\mathrm{d}^4\bar{\sigma}_{zz}}{\mathrm{d}z^4}-2a_1s^2\frac{\mathrm{d}^2\bar{\sigma}_{zz}}{\mathrm{d}z^2}+a_2s^4\bar{\sigma}_{zz}=\frac{s^2E}{1-\nu^2}\left(\alpha(1+\nu)\frac{\mathrm{d}^2\bar{T}}{\mathrm{d}z^2}-s^2\left(\alpha_z+\alpha\nu_z\frac{E}{E_z}\right)\bar{T}\right) \tag{16}$$

$$\bar{\sigma}_{zz}\Big|_{z=0}=-\bar{p},\ \frac{\mathrm{d}\bar{\sigma}_{zz}}{\mathrm{d}z}\Big|_{z=0}=-\mathrm{i}(s_x\bar{q}_x+s_y\bar{q}_y) \tag{17}$$

where

$$a_1=\frac{1}{1-\nu}\left(\frac{G}{G_z}-\nu_z\frac{E}{E_z}\right)$$
$$a_2=\frac{1}{1-\nu^2}\left(1-\nu_z^2\frac{E}{E_z}\right)\frac{E}{E_z} \tag{18}$$

Note that the elastic moduli involved in expressions (18) meet the following physical constraints[25]

$$-1<\nu<1/2,\quad \nu E_z<E_z-2\nu_z^2E$$
$$|\nu_z|<\begin{cases}1 & E_z\geqslant E\\ \sqrt{E_z/E} & E_z<E\end{cases} \tag{19}$$
$$E>0,E_z>0,G>0,G_z>0$$

The form of a solution to Eq.(16) along with conditions (17) on the limiting plane $z=0$ and the decreasing condition at the points of infinity $z\to+\infty$, strongly depends on the interrelations between the coefficients (18), as the eigenvalues of Eq.(16) can be given in the following form

$$\lambda_j=\sqrt{a_1+(-1)^j\sqrt{a_1^2-a_2}}\quad j=1,2 \tag{20}$$

In the context of constraints (19) and expressions (18), we can conclude that $a_2>0$ for all physically allowed transversely isotropic moduli so that the eigenvalues (20) can be: (A) real and dissimilar (for $a_1^2>a_2$ and $a_1>0$), (B) real and multiple (for $a_1^2=a_2$ and $a_1>0$, which is the case, e.g., of isotropic materials), (C) imaginary multiple (for $a_1^2=a_2$ and $a_1<0$), and (D) complex and dissimilar. Note that if in the latter case the eigenvalues (20) were represented in the following form

$$\lambda_j=l_j+\mathrm{i}\ell_j,l_j=\mathrm{Re}\lambda_j\neq0,\ell_j=\mathrm{Im}\lambda_j\neq0 \tag{21}$$

Then due to the obvious equalities $2a_1=\lambda_1^2+\lambda_2^2\in\mathbf{R}$ and $a_2=\lambda_1^2\lambda_2^2\in\mathbf{R}$, we necessarily conclude that $a_1=l_1^2-\ell_1^2=l_2^2-\ell_2^2$, $a_1^2-a_2=4l_1l_2\ell_1\ell_2$, and

$$l_1=-l_2,\ell_1=\ell_2 \tag{22}$$

The complete analytical solution to the formulated thermoelasticity problem is, obviously, to cover all of the foregoing cases A—D of the eigenvalues which results, particularly, in the character of the solution's asymptotic behavior at the points infinity.

For example, in case A of dissimilar real eigenvalues (20), an analytical solution to Eq.(16), which meets boundary conditions (17) and is limited at $z\to+\infty$, can be given in the following form

$$\begin{aligned}\bar{\sigma}_{zz}(z)=&\frac{1}{|\lambda_2|-|\lambda_1|}\Bigg((|\lambda_1|\mathrm{e}^{-|s\lambda_2|z}-|\lambda_2|\mathrm{e}^{-|s\lambda_1|z})\bar{p}+\\&\frac{\mathrm{i}}{|s|}(\mathrm{e}^{-|s\lambda_2|z}-\mathrm{e}^{-|s\lambda_1|z})(s_x\bar{q}_x+s_y\bar{q}_y)+\\&\Bigg(\int_0^{+\infty}\bar{T}(\zeta)(|\lambda_2|+|\lambda_1|)(\alpha_2\mathrm{e}^{-|s\lambda_2|(z+\zeta)}+\\&\alpha_1\mathrm{e}^{-|s\lambda_1|(z+\zeta)})-2\alpha_2|\lambda_2|\mathrm{e}^{-|s|(|\lambda_1|z+|\lambda_2|\zeta)}-\\&2\alpha_1|\lambda_1|\mathrm{e}^{-|s|(|\lambda_2|z+|\lambda_1|\zeta)}+(|\lambda_2|-|\lambda_1|)(\alpha_2\mathrm{e}^{-|s\lambda_2(z-\zeta)|}-\\&\alpha_1\mathrm{e}^{-|s\lambda_1(z-\zeta)|})\Bigg)\mathrm{d}\zeta\Bigg)\end{aligned} \tag{23}$$

where $\lambda_j\in\mathbf{R}$ are the eigenvalues computed by Eq.(20) at $a_1^2>a_2$ and $a_1>0$, and

$$\alpha_j=\frac{1}{2}\frac{\alpha E}{1-\nu}\frac{1}{\lambda_1^2-\lambda_2^2}\left|\frac{s}{\lambda_j}\right|\left(\lambda_j^2-\frac{\alpha_zE_z+\alpha\nu_zE}{\alpha E_z(1+\nu)}\right)$$

where $j=1,2$.

To determine the normal stress $\bar{\sigma}_{yy}$ in the mapping domain of transform (9), we use Eq.(13), which takes the following form

$$\begin{aligned}\frac{\mathrm{d}^2\bar{\sigma}_{yy}}{\mathrm{d}z^2}-(sk)^2\bar{\sigma}_{yy}=&\frac{\mu_2-\mu^+}{\mu^+\mu_1}\frac{\mathrm{d}^2\bar{\sigma}_{zz}}{\mathrm{d}z^2}+\frac{\mu_1-\mu_2}{\mu_1}\frac{\mathrm{d}^2\bar{\sigma}}{\mathrm{d}z^2}-\\&\frac{1}{\mu_1}\left(s_y^2(1-\mu_3)+s^2\frac{\mu_2-\mu^+}{\mu^+}\right)\bar{\sigma}_{zz}-\frac{1}{\mu_1}\Bigg(s_x^2(1-\\&\mu_2)+s_y^2\frac{\mu_2}{\mu^-}\Bigg)\bar{\sigma}-2G\left(\alpha\left(\frac{\mathrm{d}^2\bar{T}}{\mathrm{d}z^2}-s^2\bar{T}\right)-\alpha's_y^2\bar{T}\right)\end{aligned} \tag{24}$$

where $k=\sqrt{G/G_z}>0$.

The total stress $\bar{\sigma}$ can be eliminated from the latter equation by making use of Eq.(12), which takes the following form

$$\bar{\sigma}=\frac{1}{\mu^+}\bar{\sigma}_{zz}-\frac{1+\nu}{s^2}\frac{\mathrm{d}^2\bar{\sigma}_{zz}}{\mathrm{d}z^2}-\alpha E\bar{T} \tag{25}$$

in the mapping domain of transform (9). By differentiating the latter equation twice and making use of Eq.(16), we can obtain the following

$$\frac{\mathrm{d}^2\bar{\sigma}}{\mathrm{d}z^2}=\left(\frac{1}{\mu^+}-2(1+\nu)a_1\right)\frac{\mathrm{d}^2\bar{\sigma}_{zz}}{\mathrm{d}z^2}+s^2(1+\nu)a_2\bar{\sigma}_{zz}-2\frac{\alpha E}{1-\nu}\frac{\mathrm{d}^2\bar{T}}{\mathrm{d}z^2}+s^2\frac{E}{1-\nu}\left(\alpha_z+\alpha\nu_z\frac{E}{E_z}\right)\bar{T} \quad (26)$$

Now, putting Eqs.(25,26) into Eq.(24) yields

$$\frac{\mathrm{d}^2\bar{\sigma}_{yy}}{\mathrm{d}z^2}-(sk)^2\bar{\sigma}_{yy}=\frac{d_1}{s^2}\frac{\mathrm{d}^2\bar{\sigma}_{zz}}{\mathrm{d}z^2}+d_2\bar{\sigma}_{zz}-\frac{\alpha E}{1-\nu}\frac{\mathrm{d}^2\bar{T}}{\mathrm{d}z^2}-d_0\bar{T} \quad (27)$$

where

$$d_0=E\left(\alpha_z+\alpha\nu_z\frac{E}{E_z}\right)\frac{s_y^2+\nu s_x^2}{1-\nu^2}+\alpha E s_x^2\frac{G}{G_z}$$

$$d_1=\frac{E}{2}\left(\frac{s^2}{1-\nu}\left(\frac{\nu_z}{G}\frac{E}{E_z}-\frac{1}{G_z}\right)+\frac{s_x^2}{G_z}+2s_y^2\frac{\nu_z}{E_z}\right)$$

$$d_2=\frac{E}{E_z}\left(\left(1-\nu_z^2\frac{E}{E_z}\right)\frac{s_y^2+\nu s_x^2}{1-\nu^2}-s_x^2\nu_z\frac{G}{G_z}\right)$$

An analytical solution to Eq.(27) can be given in the following form

$$\bar{\sigma}_{yy}(z)=A\mathrm{e}^{-|s|kz}+\frac{d_1}{s^2}\bar{\sigma}_{zz}(z)-\frac{\alpha E}{1-\nu}\bar{T}(z)+\int_0^{+\infty}(c_y\bar{\sigma}_{zz}(\zeta)+d_y\bar{T}(\zeta))\mathrm{e}^{-|s|k|z-\zeta|}\mathrm{d}\zeta \quad (28)$$

where

$$c_y=-\frac{d_2+k^2d_1}{2|s|k},\quad d_y=\frac{|s|k}{2}\frac{\alpha E}{1-\nu}-\frac{d_0}{2|s|k}$$

Substituting Eq.(23) into Eq.(28) yields the following

$$\begin{aligned}\bar{\sigma}_{yy}(z)=&A\mathrm{e}^{-|s|kz}-\frac{\alpha E}{1-\nu}\bar{T}(z)+\\&\frac{1}{|\lambda_2|-|\lambda_1|}\Bigg(|\lambda_1|(\gamma_2\mathrm{e}^{-|s\lambda_2|z}-|\lambda_2|\gamma_1\mathrm{e}^{-|s\lambda_1|z}+\gamma_{0p}\mathrm{e}^{-|s|kz})\bar{p}+\\&\frac{\mathrm{i}}{|s|}(\gamma_{oq}\mathrm{e}^{-|s|kz}+\gamma_2\mathrm{e}^{-|s\lambda_2|z}-\gamma_1\mathrm{e}^{-|s\lambda_1|z})(s_x\bar{q}_x+s_y\bar{q}_y)+\\&\int_0^{+\infty}\bar{T}(\zeta)((|\lambda_2|+|\lambda_1|)(\alpha_2\gamma_2\mathrm{e}^{-|s\lambda_2|(z+\zeta)}+\\&\alpha_1\gamma_1\mathrm{e}^{-|s\lambda_1|(z+\zeta)})-2\alpha_2\gamma_1|\lambda_2|\mathrm{e}^{-|s|(|\lambda_1|z+|\lambda_2|\zeta)}-\\&2\alpha_1\gamma_2|\lambda_1|\mathrm{e}^{-|s|(|\lambda_2|z+|\lambda_1|\zeta)}+(|\lambda_2|-|\lambda_1|)\times\\&(\alpha_2\gamma_2\mathrm{e}^{-|s\lambda_2(z-\zeta)|}-\alpha_1\gamma_1\mathrm{e}^{-|s\lambda_1(z-\zeta)|})+\gamma_{0k}\mathrm{e}^{-|sk(z-\zeta)|}+\\&\gamma_{k1}\mathrm{e}^{-|s|(kz+|\lambda_1|\zeta)}+\gamma_{k2}\mathrm{e}^{-|s|(kz+|\lambda_2|\zeta)})\mathrm{d}\zeta\Bigg)\end{aligned} \quad (29)$$

where

$$\gamma_j=\frac{d_1}{s^2}+\frac{2kc_y}{|s|(k^2-\lambda_j^2)}\quad j=1,2$$

$$\gamma_{0p}=\frac{c_y}{|s|}\left(\frac{|\lambda_2|}{k-|\lambda_1|}-\frac{|\lambda_1|}{k-|\lambda_2|}\right)$$

$$\gamma_{0q}=\frac{c_y}{|s|}\left(\frac{1}{k-|\lambda_1|}-\frac{1}{k-|\lambda_2|}\right)$$

$$\gamma_{0k}=d_y+2\frac{|\lambda_2|-|\lambda_1|}{|s|}\left(\frac{\alpha_1|\lambda_1|}{k^2-\lambda_1^2}-\frac{\alpha_2|\lambda_2|}{k^2-\lambda_2^2}\right)c_y$$

$$\gamma_{k1}=2\alpha_1c_y\left|\frac{\lambda_1}{s}\right|\frac{\lambda_2^2-\lambda_1^2}{(k-|\lambda_2|)(k^2-\lambda_1^2)}$$

$$\gamma_{k2}=2\alpha_2c_y\left|\frac{\lambda_2}{s}\right|\frac{\lambda_1^2-\lambda_2^2}{(k-|\lambda_1|)(k^2-\lambda_2^2)}$$

and A is an arbitrary constant of integration. The latter one can be determined by means of condition (8), which takes the following form

$$\frac{\mathrm{d}}{\mathrm{d}z}(s_y^2\bar{\varepsilon}_{xx}(z)-s_x^2\bar{\varepsilon}_{yy}(z))\Big|_{z=0}=\mathrm{i}s_xs_y(s_y\bar{\varepsilon}_{xz}(0)-s_x\bar{\varepsilon}_{yz}(0))$$

in the mapping domain of transform (9). Making use of Eq.(3) along with Eq.(15) and boundary conditions (6) and (7) yields

$$\frac{\mathrm{d}\bar{\sigma}_{yy}(z)}{\mathrm{d}z}\Big|_{z=0}-\frac{s_y^2+\nu s_x^2}{s^2(1+\nu)}\frac{\mathrm{d}\bar{\sigma}(z)}{\mathrm{d}z}\Big|_{z=0}=\mathrm{i}(s_xq^-\bar{q}_x+s_yq^+\bar{q}_y)+\frac{s_y^2-s_x^2}{s^2}\frac{\alpha E}{1+\nu}\frac{\mathrm{d}\bar{T}(z)}{\mathrm{d}z}\Big|_{z=0} \quad (30)$$

where

$$q^+=\frac{1}{s^2(1+\nu)}\left(s_x^2\left(\nu+\frac{E}{E_z}\right)+s_y^2+(s_y^2-s_x^2)\nu_z\frac{E}{E_z}\right)$$

$$q^-=\frac{1}{s^2(1+\nu)}\left(\nu s_x^2+s_y^2\left(1-\frac{E}{E_z}\right)+(s_y^2-s_x^2)\nu_z\frac{E}{E_z}\right)$$

To derive an expression for the total stress $\bar{\sigma}(z)$ appearing in condition (30), we substitute Eq.(23) into Eq.(25), which yields

$$\begin{aligned}\bar{\sigma}(z)=&\frac{1}{|\lambda_2|-|\lambda_1|}\Bigg((|\lambda_1|\beta_2\mathrm{e}^{-|s\lambda_2|z}-|\lambda_2|\beta_1\mathrm{e}^{-|s\lambda_1|z})\bar{p}+\\&\frac{\mathrm{i}}{|s|}(\beta_2\mathrm{e}^{-|s\lambda_2|z}-\beta_1\mathrm{e}^{-|s\lambda_1|z})(s_x\bar{q}_x+s_y\bar{q}_y)+\\&\int_0^{+\infty}\bar{T}(\zeta)((|\lambda_2|+|\lambda_1|)(\alpha_2\beta_2\mathrm{e}^{-|s\lambda_2|(z+\zeta)}+\\&\alpha_1\beta_1\mathrm{e}^{-|s\lambda_1|(z+\zeta)})-2\alpha_2\beta_1|\lambda_2|\mathrm{e}^{-|s|(|\lambda_1|z+|\lambda_2|\zeta)}-\\&2\alpha_1\beta_2|\lambda_1|\mathrm{e}^{-|s|(|\lambda_2|z+|\lambda_1|\zeta)}+(|\lambda_2|-|\lambda_1|)\times\\&(\alpha_2\beta_2\mathrm{e}^{-|s\lambda_2(z-\zeta)|}-\alpha_1\beta_1\mathrm{e}^{-|s\lambda_1(z-\zeta)|}))\mathrm{d}\zeta\Bigg)-\\&2\frac{\alpha E}{1-\nu}\bar{T}(z)\end{aligned} \quad (31)$$

where

$$\beta_j = 1 + \nu_z \frac{E}{E_z} - (1+\nu)\lambda_j^2 \quad j = 1, 2$$

Now, using Eqs.(29,31) together with condition (30) allows for eliminating constant A. As a result, the stress $\bar{\sigma}_{yy}$ can be given in the following form

$$\bar{\sigma}_{yy}(z) = -\frac{\alpha E}{1-\nu}\bar{T}(z) + \frac{1}{|\lambda_2| - |\lambda_1|}\Bigg((|\lambda_1|\gamma_2 e^{-|s\lambda_2|z} - |\lambda_2|\gamma_1 e^{-|s\lambda_1|z} + (\gamma_{0p} + \gamma_{0p}^*)e^{-|s|kz})\bar{p} + \frac{is_x}{|s|}((\gamma_{0q} + \gamma_-^*)e^{-|s|kz} + \gamma_2 e^{-|s\lambda_2|z} - \gamma_1 e^{-|s\lambda_1|z})\bar{q}_x + \frac{is_y}{|s|}((\gamma_{0q} + \gamma_+^*)e^{-|s|kz} + \gamma_2 e^{-|s\lambda_2|z} - \gamma_1 e^{-|s\lambda_1|z})\bar{q}_y + \int_0^{+\infty}\bar{T}(\zeta)((|\lambda_2| + |\lambda_1|)(\alpha_2\gamma_2 e^{-|s\lambda_2|(z+\zeta)} + \alpha_1\gamma_1 e^{-|s\lambda_1|(z+\zeta)}) - 2\alpha_2\gamma_1|\lambda_2|e^{-|s|(|\lambda_1|z + |\lambda_2|\zeta)} - 2\alpha_1\gamma_2|\lambda_1|e^{-|s|(|\lambda_2|z + |\lambda_1|\zeta)} + (|\lambda_2| - |\lambda_1|)(\alpha_2\gamma_2 e^{-|s\lambda_2(z-\zeta)|} - \alpha_1\gamma_1 e^{-|s\lambda_1(z-\zeta)|}) + \gamma_{0k}(e^{-|sk(z-\zeta)|} + e^{-|s|k(z+\zeta)}) + (\gamma_{k1} + \gamma_{k1}^*)e^{-|s|(kz + |\lambda_1|\zeta)} + (\gamma_{k2} + \gamma_{k2}^*)e^{-|s|(kz + |\lambda_2|\zeta)})d\zeta\Bigg) \tag{32}$$

where

$$\gamma_{0p}^* = \frac{|\lambda_1\lambda_2|}{k}\left(\gamma_1 - \gamma_2 + \frac{s_y^2 + \nu s_x^2}{s^2}(\lambda_1^2 - \lambda_2^2)\right) - \gamma_{0p}$$

$$\gamma_\pm^* = \frac{|\lambda_1|}{k}\left(\gamma_1 - \frac{s_y^2 + \nu s_x^2}{s^2(1+\nu)}\beta_1\right) - \frac{|\lambda_2|}{k}\left(\gamma_2 - \frac{s_y^2 + \nu s_x^2}{s^2(1+\nu)}\beta_2\right) + \frac{|\lambda_1| - |\lambda_2|}{k}q^\pm - \gamma_{0q}$$

$$\gamma_{kj}^* = \alpha_j\frac{|\lambda_1\lambda_2|}{k}(-1)^j\left(\frac{s_y^2 + \nu s_x^2}{s^2}(\lambda_1^2 - \lambda_2^2), - \gamma_2 + \gamma_1\right) - \gamma_{kj} \qquad j = 1, 2$$

In such a manner, the transversal and normal stresses, $\bar{\sigma}_{zz}(z)$ and $\bar{\sigma}_{yy}(z)$, are found in the mapping domain of transform (9) in Eqs.(23,32), respectively. The stress $\bar{\sigma}_{xx}(z)$ can be found in a similar form by making use of formula (A1) presented in Appendix.

In order to derive the tangential stress-tensor components, we use Eq.(14), which can be presented as

$$\bar{\sigma}_{xy}(z) = \frac{-1}{2s_x s_y}\left(s_x^2\bar{\sigma}_{xx}(z) + s_y^2\bar{\sigma}_{yy}(z) + \frac{d^2\bar{\sigma}_{zz}(z)}{dz^2}\right) \tag{33}$$

$$\frac{d\bar{\sigma}_{yz}(z)}{dz} = \frac{i}{2s_y}\left(s_x^2\bar{\sigma}_{xx}(z) - s_y^2\bar{\sigma}_{yy}(z) + \frac{d^2\bar{\sigma}_{zz}(z)}{dz^2}\right)$$
$$x \rightleftarrows y \tag{34}$$

Making use of Eqs. (33, 34) along with Eqs.(15,25) yields the expressions of the tangential stresses in terms of the key ones in the forms given by formulae (A2—A4) in Appendix.

In case B of the multiple real eigenvalues (20), coefficients (18) of Eq.(16) are to meet the following conditions: $a_1^2 = a_2 = a_0^4$ and $a_1 > 0$. In view of Eq.(18), these conditions imply

$$4\nu_z(1+\nu)\left(2\frac{\nu_z}{E_z} - \frac{1}{G_z}\right) + \frac{E_z}{G_z^2} = 2\frac{1-\nu}{G} \tag{35}$$

and

$$2\nu_z(1+\nu) < \frac{E_z}{G_z}$$

Hence, a solution to Eq.(16) which meets conditions (17) and vanishes at infinitely distant points can be given in the following form

$$\bar{\sigma}_{zz}(z) = -((1 + |a_0 s|z)\bar{p} + i(s_x\bar{q}_x + s_y\bar{q}_y)z)e^{-|sa_0|z} + \int_0^{+\infty}\bar{T}(\zeta)((\alpha_1^0 + \alpha_2^0|a_0 s||z-\zeta|)e^{-|sa_0||z-\zeta|} - (\alpha_1^0 - \alpha_2^0|a_0 s|(z-\zeta) + 2|a_0 s|z(\alpha_1^0 + \alpha_2^0|a_0 s|\zeta))e^{-|sa_0|(z+\zeta)})d\zeta \tag{36}$$

where

$$\alpha_j^0 = \frac{G}{2(1-\nu)}\left|\frac{s}{a_0^3}\right|\left(\alpha\left((-1)^j a_0^2(1+\nu) - \nu_z\frac{E}{E_z}\right) - \alpha_z\right)$$

and $j = 1, 2$.

After stress $\bar{\sigma}_{zz}(z)$ is determined in Eq.(36), the total stress for case B can be found from Eq.(25); stresses $\bar{\sigma}_{yy}(z)$ and $\bar{\sigma}_{xx}(z)$ are then to be subsequently determined from Eq.(28) along with condition (30) and Eq.(A1), respectively; then the tangential stresses can be computed by means of Eqs.(A2—A4).

In case C of the multiple imaginary eigenvalues (20), coefficients (18) meet the conditions $a_1^2 = a_2 = a^4$ and $a_1 < 0$. Then Eq.(35) holds, while

$$2\nu_z(1+\nu) > \frac{E_z}{G_z} \tag{37}$$

Although Eq.(37) may seem unrealistic, there is however a practical possibility of the existence of such materials in view of Eq.(19).

A solution to Eq.(16) in this case can be given as

$$\bar{\sigma}_{zz}(z)=(C_1+zC_2)\sin(saz)+(C_3+zC_4)\cos(saz)+\int_0^{+\infty}\bar{T}(\zeta)(\alpha^-\sin(as|z-\zeta|)+\alpha^+as|z-\zeta|\cos(sa(z-\zeta)))\mathrm{d}\zeta \quad (38)$$

where

$$\alpha^{\pm}=\frac{sG}{2a^3(1-\nu)}\left(\alpha\left(a^2(1+\nu)\pm\nu_z\frac{E}{E_z}\right)\pm\alpha_z\right)$$

The constants of integration $C_j, j=\overline{1,4}$ in Eq.(38) can be eliminated by making use of the vanishing conditions at the points of infinity and the boundary conditions (17). As a result, the stress takes the following form

$$\bar{\sigma}_{zz}(z)=-\bar{p}\cos(saz)+\mathrm{i}(s_x\bar{q}_x+s_y\bar{q}_y)\frac{\sin(saz)}{sa}+\int_0^{+\infty}\bar{T}(\zeta)(\alpha^-(\sin(as|z-\zeta|)+\sin(as(z-\zeta)))+\alpha^+(as(|z-\zeta|-z-\zeta)\cos(sa(z-\zeta))+\sin(sa(z+\zeta))))\mathrm{d}\zeta$$

In case D of the dissimilar complex eigenvalues (20), their real l_j and imaginary ℓ_j parts indicated in Eq.(21), meet necessary conditions (22). By denoting $\lambda=|l_j|$ and $\mu=|\ell_j|$, we can construct the vanishing solution to Eq.(16) with conditions (17) in the form as follows

$$\bar{\sigma}_{zz}(z)=-\left((\mu\cos(\mu|s|z)+\lambda\sin(\mu|s|z))\bar{p}+\frac{\mathrm{i}}{s}(s_x\bar{q}_x+s_y\bar{q}_y)\sin(\mu|s|z)\right)\frac{\mathrm{e}^{-\lambda|s|z}}{\mu}+\int_0^{+\infty}\bar{T}(\zeta)((f_c(\zeta)\cos(\mu|s|z)+f_s(\zeta)\sin(\mu|s|z))\mathrm{e}^{-\lambda|s|(z+\zeta)}+(\lambda\alpha_c^+\sin(\mu|s||z-\zeta|)+\mu\alpha_c^-\cos(\mu|s|(z-\zeta)))\mathrm{e}^{-\lambda|s||z-\zeta|})\mathrm{d}\zeta$$

where

$$f_c(\zeta)=-\lambda\alpha_c^+\sin(\mu|s|\zeta)-\mu\alpha_c^-\cos(\mu|s|\zeta)$$

$$f_s(\zeta)=\lambda(\alpha_c^+-2\alpha_c^-)\cos(\mu|s|\zeta)-\frac{\mu^2\alpha_c^-+2\lambda^2\alpha_c^+}{\mu}\sin(\mu|s|\zeta)$$

and

$$\alpha_c^{\pm}=\frac{|s|G}{2\lambda\mu(\lambda^2+\mu^2)(1-\nu)}\left(\alpha\left(\pm(1+\nu)\times(\lambda^2+\mu^2)-\nu_z\frac{E}{E_z}\right)-\alpha_z\right)$$

Then the rest of the stresses for cases C and D can be found by the routine used for the foregoing cases A and B.

After the stress-tensor components are found in the mapping domain of the Fourier integral transform (9), they can be restored in the physical domain by making use of the inverse transform

$$f(x,y,z)=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\bar{f}(z)\mathrm{e}^{\mathrm{i}(xs_x+ys_y)}\mathrm{d}s_x\mathrm{d}s_y \quad (39)$$

realized either numerically or by an analytical mean. Similarly, the physical value of the temperature can be computed by applying Eq.(39) to Eq.(10).

3 Numerical Examples and Discussion

To verify the efficiency of the constructed solution, we compute the stresses in various transversely isotropic materials (Table 1) under the local normal force loading Eq.(6) of the following profile

$$p=p_0=\begin{cases}1 & (x,y)\in[-1,1]^2\\ 0 & \text{else}\end{cases} \qquad q_x=q_y=0 \quad (40)$$

where p_0 is a constant in the dimension of stresses. Note that the properties of materials presented in rows 1—3 of Table 1 correspond to case A, while the material in the 4th row corresponds to case D.

Table 1 Elastic moduli of the considered transversely isotropic materials[16, 26]

Material	E/ GPa	E_z/ GPa	ν	ν_z	G/ GPa	G_z/ GPa
Carbon fiber	15.00	232.00	0.49	0.28	5.03	24.00
Ceramic PZT-4	81.28	64.53	0.33	0.34	30.56	25.60
Composite 60% fiber	9.95	141.10	0.50	0.27	3.32	6.00
Hexagonal zinc	13.56	5.04	0.21	0.17	5.60	3.85

Fig.1 presents the full-field analysis of the stress σ_{zz} normalized by the parameter p_0 at the cross-section $x=0$. As we can observe, the computed stress exactly satisfies the boundary conditions (6, 40), and vanishes at a distance from the

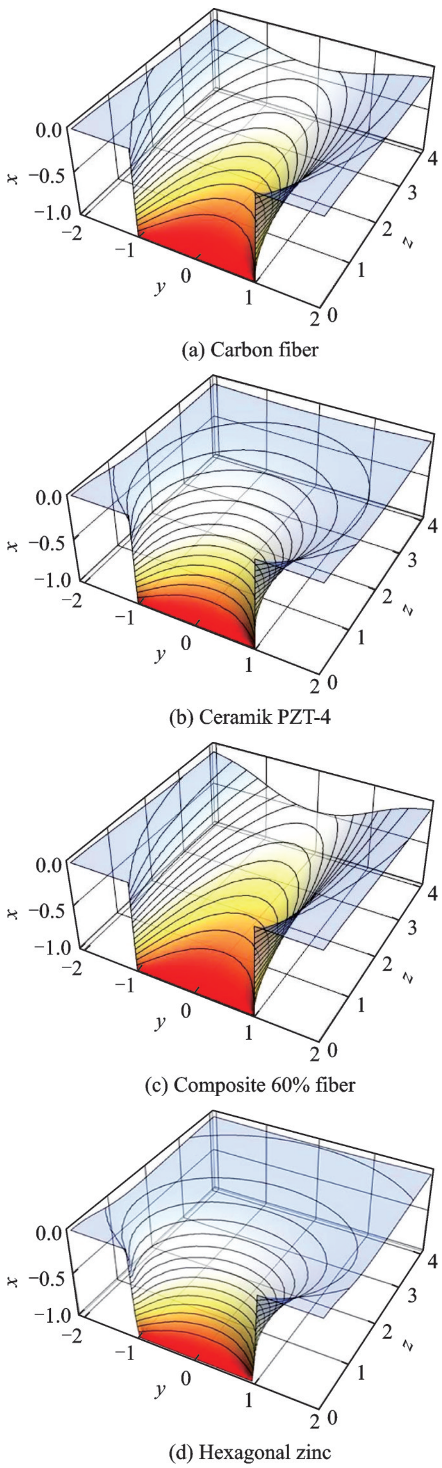


Fig.1 Full-field distributions of normal stress σ_{zz}/p_0 at $x=0$ for different transversely isotropic materials under loading (40)

loaded segment of the surface $z=0$. It is also notable that the material properties play a crucial role in the quantitative behavior of the stress.

The effect of material properties in the stress distribution is clearly pronounced in Fig.2 presenting the same stress under the center of the loaded

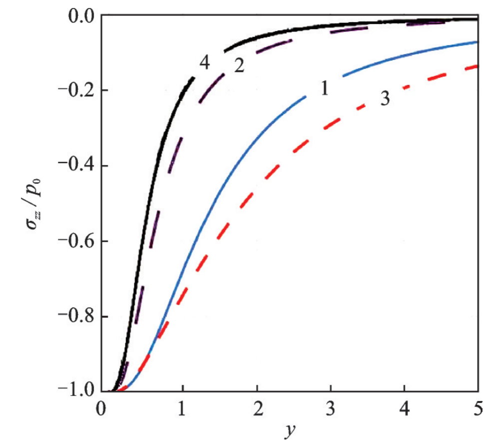


Fig.2 Normal stress σ_{zz}/p_0 at $x=0$, $y=0$ for carbon fiber (curve 1), ceramic PZT-4 (curve 2), composite 60% fiber (curve 3), and hexagonal zinc (curve 4)

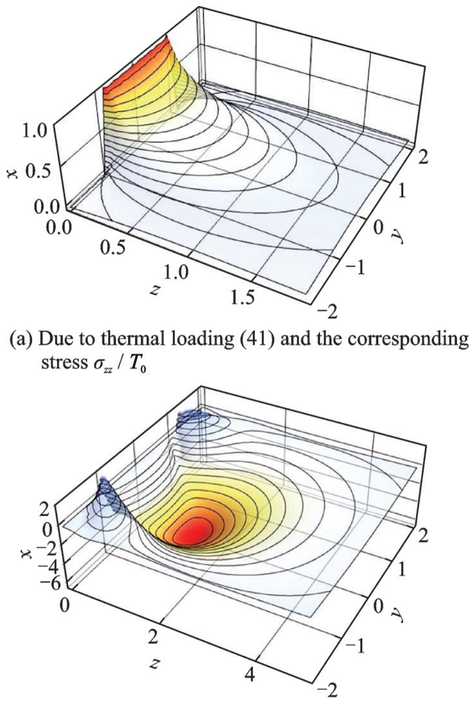
zone: $x=y=0$. It is also notable that the curves presented in this figure are orthogonal (in the meaning of differential geometry) to the surface at $z=0$, which in view of the equilibrium Eq.(1) indicates the zero boundary conditions for the tangential stress. The latter conclusion agrees with the boundary conditions (40) and Eq.(7).

Consider the results of computation of thermal stresses in a transversely isotropic half-space made of hexagonal zinc (Table 1, the 4th row), for which[16]: $c=c_z=124\,[\mathrm{W/(K\cdot m)}]$, $\alpha=5.818\times10^{-6}\,[1/\mathrm{K}]$, and $\alpha_z=15.350\times10^{-6}\,[1/\mathrm{K}]$. Assume the surface $z=0$ of the half-space to be free of force loadings, i.e. $p=q_x=q_y=0$. Instead, the surface undergoes the thermal impact (5), where

$$T=T_0=\begin{cases}1 & (x,y)\in[-1,1]^2\\ 0 & \text{else}\end{cases}\qquad a=1, b=0 \quad (41)$$

and T_0 is a constant parameter in the dimension of temperature.

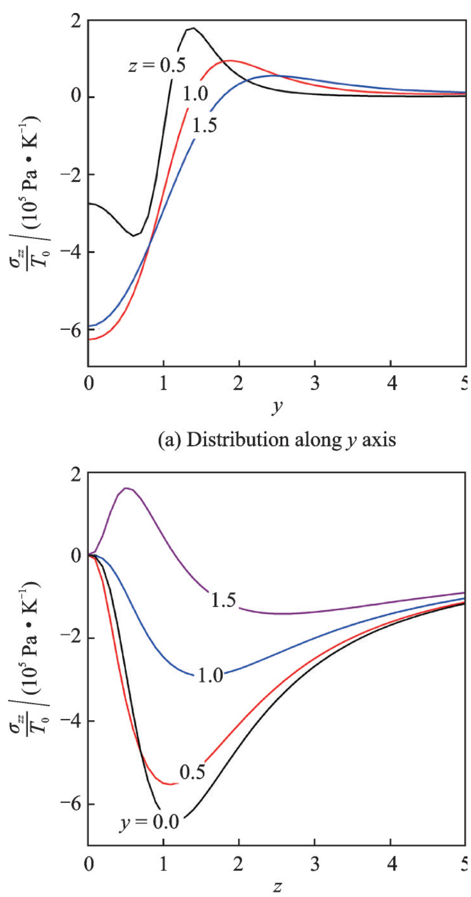
Fig.3 presents the full-field distribution of temperature (10) in the physical domain of inverse transform (39) under condition (41) and the corresponding thermal stress σ_{zz}. For the considered steady-state case, the disturbance of temperature occurs in the vicinity of the heated zone of the surface and decreases rapidly when moving away from it. The stress meets homogeneous boundary condition and is also locally disturbed over the area under the heated segment of the surface. In Fig.4, stress σ_{zz} is shown in some characteristic cross-sections of



(a) Due to thermal loading (41) and the corresponding stress σ_{zz}/T_0

(b) In the cross-section $x = 0$

Fig.3 Full-field distributions of the dimensionless temperature T/T_0 under different conditions



(a) Distribution along y axis

(b) Distribution along z axis

Fig.4 Distributions of the stress σ_{zz}/T_0 in the characteristic cross-sections of the thermally loaded half-space ($x = 0$)

the half-space in the vicinity of the thermally affected area.

4 Conclusions

This paper presents an analytical technique for exact thermoelastic analysis of a transversely isotropic half-space subject to local thermal and force loadings. The technique is based on the application of the direct integration method which allows for the reduction of the original thermoelasticity problem to a set of governing equations for the individual stress-tensor components. The equations are accompanied by the corresponding local and integral boundary conditions.

Making use of the proposed solution technique allows for capturing explicit dependencies between the applied thermal and force impacts and the induced stress field for any possible case of interrelations between the effective moduli of transversely isotropic material. Special attention is given to the correct asymptotic of the constructed solutions when moving away from the zones where the loadings were applied.

The constructed solutions can be used for the analysis of thermal and force impacts on the elastic semi-infinite composites made of materials exhibiting transversal isotropic properties. Due to its explicit form, it may serve as an efficient tool in solving inverse problems[19], as well as the verification of results gained by either numerical or semi-analytical means.

Appendix Formulae for computation of the normal and tangential stresses by the known key stresses

The normal stress

$$\bar{\sigma}_{xx}(z)=\bar{\sigma}(z)-\bar{\sigma}_{yy}(z)-\bar{\sigma}_{zz}(z) \tag{A1}$$

The tangential stresses

$$\bar{\sigma}_{xy}(z)=\frac{1}{2s_x s_y}\left(\frac{s_y^2-\nu s_x^2}{1+\nu}\bar{\sigma}(z)+(s_x^2-s_y^2)\bar{\sigma}_{yy}(z)+ s^2\frac{\alpha E}{1+\nu}\bar{T}(z)-\left(s^2\nu_z\frac{2G}{E_z}+\frac{s_y^2-\nu s_x^2}{1+\nu}\right)\bar{\sigma}_{zz}(z)\right) \tag{A2}$$

$$\bar{\sigma}_{yz}(z)=\frac{\bar{q}_y}{2}+\frac{\mathrm{i}}{4s_y}\int_0^{+\infty}\left(\frac{\nu s_x^2-s_y^2}{1+\nu}\bar{\sigma}(\zeta)-s^2\bar{\sigma}_{yy}(\zeta)-\frac{s^2\alpha E}{1+\nu}\bar{T}(\zeta)+\left(s^2\nu_z\frac{2G}{E_z}+\frac{s_y^2-\nu s_x^2}{1+\nu}\right)\bar{\sigma}_{zz}(\zeta)\right)\mathrm{sgn}(z-\zeta)\mathrm{d}\zeta$$

(A3)

$$\bar{\sigma}_{xz}(z)=\frac{\bar{q}_x}{2}+\frac{\mathrm{i}}{4s_x}\int_0^{+\infty}\left(s^2\left(\bar{\sigma}_{yy}(\zeta)-\frac{\alpha E}{1+\nu}\bar{T}(\zeta)\right)+\left(s^2\nu_z\frac{2G}{E_z}+\frac{(2+\nu)s_x^2+s_y^2}{1+\nu}\right)\bar{\sigma}_{zz}(\zeta)-\frac{(2+\nu)s_x^2+s_y^2}{1+\nu}\bar{\sigma}(\zeta)\right)\mathrm{sgn}(z-\zeta)\mathrm{d}\zeta \quad (A4)$$

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半平面横观各向同性热弹体三维热应力分析

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摘要:利用直接积分方法,求解了横观各向同性热弹材料在力-热载荷作用下的三维半平面问题。其中,材料的各向同性对称面与半无限平面平行。本文将原始热弹方程简化为仅含应力张量的控制方程,且考虑了远端应力有限性等条件已确保控制方程的正确性。计算结果显式地呈现了机械和热学载荷对热弹体的影响并满足固体连续性条件。本文给出的求解方法适用于各类半无限热弹体力-热加载问题,并可为其他求解方法提供参考。

关键词:三维问题;解析解;横观各向同性复合材料;半无限模型;力学-热学加载;有限应力分布