Fractional Birkhoffian Dynamics Based on Quasi-fractional Dynamics Models and Its Lie Symmetry

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Abstract: In order to investigate the dynamic behavior of non-conservative systems, the Lie symmetries and conserved quantities of fractional Birkhoffian dynamics based on quasi-fractional dynamics model are proposed and studied. The quasi-fractional dynamics model here refers to the variational problem based on the definition of Riemann-Liouville fractional integral (RLFI), the variational problem based on the definition of extended exponentially fractional integral (EEFI), and the variational problem based on the definition of fractional integral extended by periodic laws (FIEPL). First, the fractional Pfaff-Birkhoff principles based on quasi-fractional dynamics models are established, and the corresponding Birkhoff's equations and the determining equations of Lie symmetry are obtained. Second, for fractional Birkhoffian systems based on quasi-fractional models, the conditions and forms of conserved quantities are given, and Lie symmetry theorems are proved. The Pfaff-Birkhoff principles, Birkhoff's equations and Lie symmetry theorems of quasi-fractional Birkhoffian systems and classical Birkhoffian systems are special cases of this article. Finally, some examples are given.

Key words: quasi-fractional dynamics model; Lie symmetry; conserved quantity; fractional Birkhoffian system; Riemann-Liouville derivative

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0 Introduction

Fractional calculus has been widely used in various fields of engineering and science[1-3] in recent years, because it is more accurate to describe the dynamic behavior and physical process of complex systems than the integral order model. They are also used in many dynamical systems in interdisciplinary fields, such as electromechanical systems[4], biomedical systems[5], mechanical systems[6-7] and mathematical systems[8]. In 1996—1997, Riewe[9-10] introduced fractional calculus into dynamics modeling of non-conservative systems and established fractional Hamilton equations and fractional Lagrange equations. For fractional variational problems, Agrawal et al., Atanacković et al., and Torres et al. have made a more in-depth study[11-16].

As a natural extension of Hamiltonian mechanics, Birkhoffian mechanics are more general than Hamiltonian mechanics. In 1927, Birkhoff introduced a more general equation than Hamilton equation[17], which was named Birkhoff's equation by Santilli[18] in 1983. After that, Santilli studied the Birkhoff's equation and its transformation theory. In 1996, Mei et al. set up the theoretical framework of Birkhoffian mechanics[19] and put forward its symmetry theory[20-22]. Due to the generality of Birkhoffian system, many scholars have introduced it into nonlinear dynamical systems[23-24], constrained dynamical systems[25] and quantum systems[26]. Moreover, fractional calculus has been introduced into Birkhoffian system by many scholars and the corresponding symmetries have been obtained[27-33].

In 1979, Lutzky[34] introduced the Lie method

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into the dynamical system, studied the invariant properties of second-order dynamical system under infinitesimal transformations of time, coordinates and velocity, and established the relation between Lie symmetry and Noether conserved quantities. Since then, Prince and Eliezer[35] extended Lutzky's research to the classical Kepler problem, and obtained the corresponding Lie symmetry. In 1994, Zhao[36] extended Lie symmetry to non-conservative dynamical systems. Lie symmetries and conserved quantities have made important progress in constrained mechanical systems, nonholonomic systems and Birkhoffian system on time scales in recent years[37-45].

In 2005, El-Nabulsi proposed a kind of non-conservative dynamics models based on the definition of Riemann-Liouville fractional integration (RLFI)[46], and further put forward the dynamics models based on the definition of extended exponentially fractional integral (EEFI)[47] and based on the definition of fractional integral extended by periodic laws (FIEPL)[48]. These three models are known as quasi-fractional dynamics models or El-Nabulsi models. The dynamic equations based on these models are simple and similar to the classical conservative system's Lagrange equation. Its novelty lies in that the generalized fractional order external force corresponding to the dissipative force appears in the equations instead of the fractional order derivative, and the fractional order time integration only needs one parameter, while any number of fractional order parameters will appear in other models. After that, El-Nabulsi et al. and Frederico et al. established the equations of motion of the quasi-fractional dynamics models, and extended them to nonholonomic, holonomic and dissipative dynamical system[49-50]. In recent years, Zhang and his colleagues obtained the differential equations and symmetry theory of these models in Lagrangian system, Birkhoffian system and Hamiltonian system[51-57]. However, these studies are limited to the integer order, and the fractional order is more accurate than the integer order in describing the mechanical and physical behavior of complex systems. Therefore, in this article, we further consider the Lie symmetry for fractional Birkhoffian system with Riemann-Liouville derivatives based on quasi-fractional dynamics models.

1 Preliminaries

We list some basic properties and definitions of fractional derivatives. For more detailed discussion and proof, please refer to Ref.[3].

If functions $f(t)$ and $g(t)$ are integrable and continuous on interval $[a,b]$, the definitions of Riemann-Liouville derivative of order α is

$$ {}_aD_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-\tau)^{-\alpha} f(\tau)\,\mathrm{d}\tau \tag{1}$$

$$ {}_tD_b^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)\int_t^b (\tau-t)^{-\alpha} f(\tau)\,\mathrm{d}\tau \tag{2}$$

where $\Gamma(*)$ is the Euler-Gamma function and $0\leqslant\alpha<1$.

If ${}_aD_t^{\beta-\alpha}f(t)$ exits and $\alpha>0$, $\beta>0$, we have

$$ {}_aD_t^\beta({}_aD_t^{-\alpha}f(t))={}_aD_t^{\beta-\alpha}f(t) \tag{3}$$

For $0<\alpha<1$, $\beta>0$, we have

$$ {}_aD_t^{-\beta}({}_aD_t^\alpha f(t))={}_aD_t^{\alpha-\beta}f(t)-[{}_aD_t^{\alpha-1}f(t)]_{t=a}\frac{(t-a)^{\beta-1}}{\Gamma(\beta)} \tag{4}$$

If $f(t)$ vanishes at $t=a$ and $0<\alpha<1$, we have

$$ \frac{\mathrm{d}}{\mathrm{d}t}{}_aD_t^\alpha f(t)={}_aD_t^\alpha\frac{\mathrm{d}}{\mathrm{d}t}f(t)={}_aD_t^{\alpha+1}f(t) \tag{5}$$

If ${}_aD_t^\alpha g(t)$ and ${}_tD_b^\alpha f(t)$ are existent and continuous for $t\in[a,b]$, and $f(t)$ vanishes at $t=a$ or $g(t)$ vanishes at $t=b$, we have

$$ \int_a^b f(t)\,{}_aD_t^\alpha g(t)\,\mathrm{d}t=\int_a^b g(t)\,{}_tD_b^\alpha f(t)\,\mathrm{d}t \tag{6}$$

2 Fractional Birkhoff's Equations Under Quasi-fractional Dynamics Models

2.1 Fractional Birkhoff's equations based on RLFI

According to Ref.[46], the fractional Pfaff action based on RLFI can be expressed as

$$ S_{\mathrm{R}}=\frac{1}{\Gamma(\alpha)}\int_a^b [R_\mu(\tau,a^\nu)\,{}_aD_\tau^\beta a^\mu-B(\tau,a^\nu)](t-\tau)^{\alpha-1}\mathrm{d}\tau \tag{7}$$

where a^{μ} are Birkhoff's variables; $B=B(\tau, a^{\nu})$ is the Birkhoffian; $R_{\mu}=R_{\mu}(\tau, a^{\nu})$ are Birkhoff's functions, $0\leqslant\beta<1$; t is the observer time; τ is the intrinsic time; $\tau\neq t$, functions B and R_{μ} are C^2 functions of their variables.

The fractional Pfaff-Birkhoff variational principle based on RLFI can be expressed as

$$\delta S_{\mathrm{R}}=\frac{1}{\Gamma(\alpha)}\int_a^b[\delta R_{\mu\;a}D_{\tau}^{\beta}a^{\mu}+R_{\mu}\delta({}_aD_{\tau}^{\beta}a^{\mu})-\delta B](t-\tau)^{\alpha-1}\mathrm{d}\tau=0 \tag{8}$$

It is under the commutative conditions

$$\delta\,{}_aD_{\tau}^{\beta}a^{\mu}={}_aD_{\tau}^{\beta}\delta a^{\mu}\quad \mu=1,2,\cdots,2n \tag{9}$$

and given terminal conditions

$$a^{\mu}\big|_{\tau=a}=a_1^{\mu},a^{\mu}\big|_{\tau=b}=a_2^{\mu}\quad \mu=1,2,\cdots,2n \tag{10}$$

Using integration by parts to calculate the second term in Eq.(8) and considering Eq.(10), we have

$$\frac{1}{\Gamma(\alpha)}\int_a^b R_{\mu}\delta({}_aD_{\tau}^{\beta}a^{\mu})(t-\tau)^{\alpha-1}\mathrm{d}\tau=\frac{1}{\Gamma(\alpha)}\int_a^b R_{\mu\;a}D_{\tau}^{\beta}(\delta a^{\mu})(t-\tau)^{\alpha-1}\mathrm{d}\tau=\frac{1}{\Gamma(\alpha)}\int_a^b\delta a^{\mu}\;{}_{\tau}D_b^{\beta}[R_{\mu}(t-\tau)^{\alpha-1}]\mathrm{d}\tau \tag{11}$$

Substituting Eq.(11) into principle in Eq.(8), we get

$$\frac{1}{\Gamma(\alpha)}\int_a^b\left\{\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}\;{}_aD_{\tau}^{\beta}a^{\nu}-\frac{\partial B}{\partial a^{\mu}}\right)(t-\tau)^{\alpha-1}+{}_{\tau}D_b^{\beta}[R_{\mu}(t-\tau)^{\alpha-1}]\right\}\delta a^{\mu}\mathrm{d}\tau=0 \tag{12}$$

Since Eq.(12) is true for any integral interval $[a,b]$ and the independence of δa^{μ}, by using the fundamental lemma[58] of the calculus of variations, we have

$$\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}\;{}_aD_{\tau}^{\beta}a^{\nu}-\frac{\partial B}{\partial a^{\mu}}\right)(t-\tau)^{\alpha-1}+{}_{\tau}D_b^{\beta}[R_{\mu}(t-\tau)^{\alpha-1}]=0\quad \mu=1,2,\cdots,2n \tag{13}$$

Eq.(13) is the bundle of fractional Birkhoff's equations based on RLFI.

2.2 Fractional Birkhoff's equations based on EEFI

According to Ref.[47], the fractional Pfaff action based on EEFI can be expressed as

$$S_{\mathrm{E}}=\frac{1}{\Gamma(\alpha)}\int_a^b[R_{\mu}(\tau,a^{\nu})\;{}_aD_{\tau}^{\beta}a^{\mu}-B(\tau,a^{\nu})](\cosh t-\cosh\tau)^{\alpha-1}\mathrm{d}\tau \tag{14}$$

The fractional Pfaff-Birkhoff variational principle based on EEFI can be expressed as

$$\delta S_{\mathrm{E}}=\frac{1}{\Gamma(\alpha)}\int_a^b[\delta R_{\mu\;a}D_{\tau}^{\beta}a^{\mu}+R_{\mu}\delta({}_aD_{\tau}^{\beta}a^{\mu})-\delta B]\cdot(\cosh t-\cosh\tau)^{\alpha-1}\mathrm{d}\tau=0 \tag{15}$$

under the commutative conditions

$$\delta\,{}_aD_{\tau}^{\beta}a^{\mu}={}_aD_{\tau}^{\beta}\delta a^{\mu}\quad \mu=1,2,\cdots,2n \tag{16}$$

and given terminal conditions

$$a^{\mu}\big|_{\tau=a}=a_1^{\mu},a^{\mu}\big|_{\tau=b}=a_2^{\mu}\quad \mu=1,2,\cdots,2n \tag{17}$$

Similar to Eq.(12), from Eq.(17), we can easily get

$$\frac{1}{\Gamma(\alpha)}\int_a^b\left\{\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}\;{}_aD_{\tau}^{\beta}a^{\nu}-\frac{\partial B}{\partial a^{\mu}}\right)\cdot(\cosh t-\cosh\tau)^{\alpha-1}+{}_{\tau}D_b^{\beta}[R_{\mu}(\cosh t-\cosh\tau)^{\alpha-1}]\right\}\cdot\delta a^{\mu}\mathrm{d}\tau=0 \tag{18}$$

On account of the independence of δa^{μ} and the lemma of the calculus of variations, from Eq.(18), we obtain

$$\left(\frac{\partial R_{\nu}}{\partial a^{\mu}}\;{}_aD_{\tau}^{\beta}a^{\nu}-\frac{\partial B}{\partial a^{\mu}}\right)(\cosh t-\cosh\tau)^{\alpha-1}+{}_{\tau}D_b^{\beta}[R_{\mu}(\cosh t-\cosh\tau)^{\alpha-1}]=0\quad \mu=1,2,\cdots,2n \tag{19}$$

Eq.(19) is the bundle of fractional Birkhoff's equations based on EEFI.

2.3 Fractional Birkhoff's equations based on FIEPL

According to Ref.[48], the fractional Pfaff action based on FIEPL can be expressed as

$$S_{\mathrm{P}}=\frac{1}{\Gamma(\alpha)}\int_a^b[R_{\mu}(\tau,a^{\nu})\;{}_aD_{\tau}^{\beta}a^{\mu}-B(\tau,a^{\nu})]\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]\mathrm{d}\tau \tag{20}$$

The fractional Pfaff-Birkhoff variational principle based on FIEPL can be expressed as

$$\delta S_{\mathrm{P}}=\frac{1}{\Gamma(\alpha)}\int_a^b[\delta R_{\mu\;a}D_{\tau}^{\beta}a^{\mu}+R_{\mu}\delta({}_aD_{\tau}^{\beta}a^{\mu})-\delta B]\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]\mathrm{d}\tau=0 \tag{21}$$

under the commutative conditions

$$\delta\,{}_aD_{\tau}^{\beta}a^{\mu}={}_aD_{\tau}^{\beta}\delta a^{\mu}\quad \mu=1,2,\cdots,2n \tag{22}$$

and given terminal conditions

$$a^{\mu}\big|_{\tau=a}=a_1^{\mu},a^{\mu}\big|_{\tau=b}=a_2^{\mu}\quad \mu=1,2,\cdots,2n \tag{23}$$

Similar to Eq.(12), from Eq.(20), we can easily get

$$\frac{1}{\Gamma(\alpha)}\int_a^b\left\{\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)\cdot \sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]+{}_\tau D_b^\beta\left[R_\mu\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)\right]\right\}\cdot\delta a^\mu \mathrm{d}\tau=0 \tag{24}$$

On account of the independence of δa^μ and the lemma of the calculus of variations, from Eq.(24), we obtain

$$\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]+{}_\tau D_b^\beta\left\{R_\mu\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]\right\}=0$$
$$\mu=1,2,\cdots,2n \tag{25}$$

Eq.(25) is the bundle of fractional Birkhoff's equations based on FIEPL.

If $\beta\to 1$, Eqs.(13, 19, 25) become Birkhoff's equations based on quasi-fractional models. If $\beta\to 1$, $\alpha\to 1$, Eqs.(13, 19, 25) become the standard Birkhoff's equations[19].

3 Lie Symmetry for Fractional Birkhoffian System

3.1 Lie symmetry based on RLFI

The infinitesimal transformation of the group is introduced as

$$\bar{\tau}=\tau+\Delta\tau,\bar{a}^\mu(\bar{\tau})=a^\mu(\tau)+\Delta a^\mu$$
$$\mu=1,2,\cdots,2n \tag{26}$$

and the extension formulas are

$$\bar{\tau}=\tau+\varepsilon\xi_0(\tau,a^\nu)$$
$$\bar{a}^\mu(\bar{\tau})=a^\mu(\tau)+\varepsilon\xi_\mu(\tau,a^\nu)$$
$$\mu=1,2,\cdots,2n \tag{27}$$

where ξ_0 and ξ_μ are the infinitesimal generators; and ε is an infinitesimal parameter.

Under infinitesimal transformation in Eq.(26), there are[32]

$${}_{\bar{a}}D_{\bar{\tau}}^\beta\bar{a}^\mu(\bar{\tau})={}_aD_\tau^\beta a^\mu+{}_aD_\tau^\beta\Delta a^\mu-{}_aD_\tau^\beta(\dot{a}^\mu\Delta\tau)+\Delta\tau\,{}_aD_\tau^\beta\dot{a}^\mu\quad \mu=1,2,\cdots,2n;0\leqslant\beta<1 \tag{28}$$

The infinitesimal generator vector is introduced

$$\boldsymbol{X}^{(0)}=\xi_0\frac{\partial}{\partial\tau}+\xi_\mu\frac{\partial}{\partial a^\mu} \tag{29}$$

and its β extension[32] is

$$\boldsymbol{X}^{(\beta)}=\boldsymbol{X}^{(0)}+({}_aD_\tau^\beta\xi_\mu-{}_aD_\tau^\beta(\dot{a}^\mu\xi_0)+\xi_0\,{}_aD_\tau^\beta\dot{a}^\mu)\frac{\partial}{\partial\,{}_aD_\tau^\beta a^\mu} \tag{30}$$

Because of the invariance theory of differential equation under infinitesimal transformation, the invariance of fractional Birkhoff's equations in Eq.(13) under the infinitesimal transformation in Eq.(27) can lead to the following equations

$$\boldsymbol{X}^{(\beta)}\left\{\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)(t-\tau)^{\alpha-1}+{}_\tau D_b^\beta[R_\mu(t-\tau)^{\alpha-1}]\right\}=0$$
$$\mu=1,2,\cdots,2n \tag{31}$$

By substituting operator in Eq.(30) into Eq.(31), we get

$$\boldsymbol{X}^{(0)}\left(\frac{\partial R_\nu}{\partial a^\mu}\right){}_aD_\tau^\beta a^\nu(t-\tau)^{\alpha-1}-\boldsymbol{X}^{(0)}\left(\frac{\partial B}{\partial a^\mu}\right)(t-\tau)^{\alpha-1}+\boldsymbol{X}^{(\beta)}\left\{{}_\tau D_b^\beta[R_\mu(t-\tau)^{\alpha-1}]\right\}+\xi_0(1-\alpha)(t-\tau)^{\alpha-2}\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)+({}_aD_\tau^\beta\xi_\nu-{}_aD_\tau^\beta(\dot{a}^\nu\xi_0)+\xi_0\,{}_aD_\tau^\beta\dot{a}^\nu)\frac{\partial R_\nu}{\partial a^\mu}\cdot(t-\tau)^{\alpha-1}=0\quad \mu=1,2,\cdots,2n \tag{32}$$

Eq.(32) is the determining equation.

Definition 1 For the transformation in Eq.(27), if the determining equations in Eq.(32) are satisfied, the symmetry is the Lie symmetry of fractional Birkhoffian system Eq.(13) based on RLFI.

3.2 Lie symmetry based on EEFI

The invariance of Eq.(19) under the infinitesimal transformation in Eq.(27) can lead to the following equations

$$\boldsymbol{X}^{(\beta)}\left\{\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)\cdot(\cosh t-\cosh\tau)^{\alpha-1}+{}_\tau D_b^\beta[R_\mu(\cosh t-\cosh\tau)^{\alpha-1}]\right\}=0$$
$$\mu=1,2,\cdots,2n \tag{33}$$

By substituting operator Eq.(30) into Eq.(33), we get

$$\boldsymbol{X}^{(0)}\left(\frac{\partial R_\nu}{\partial a^\mu}\right){}_aD_\tau^\beta a^\nu(\cosh t-\cosh\tau)^{\alpha-1}-$$
$$\boldsymbol{X}^{(0)}\left(\frac{\partial B}{\partial a^\mu}\right)(\cosh t-\cosh\tau)^{\alpha-1}+$$
$$\boldsymbol{X}^{(\beta)}\left\{{}_\tau D_b^\beta[R_\mu(\cosh t-\cosh\tau)^{\alpha-1}]\right\}+$$
$$\xi_0(\alpha-1)\sinh\tau(\cosh t-\cosh\tau)^{\alpha-2}\cdot$$
$$\left(\frac{\partial R_\nu}{\partial a^\mu}{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)+({}_aD_\tau^\beta\xi_\nu-{}_aD_\tau^\beta(\dot a^\nu\xi_0)+$$
$$\xi_0\,{}_aD_\tau^\beta\dot a^\nu)\frac{\partial R_\nu}{\partial a^\mu}\cdot(\cosh t-\cosh\tau)^{\alpha-1}=0$$
$$\mu=1,2,\cdots,2n \tag{34}$$

Eq.(34) is the determining equation.

Definition 2 For the transformation in Eq.(27), if the determining equations in Eq.(34) are satisfied, the symmetry is the Lie symmetry of fractional Birkhoffian system in Eq.(19) based on EEFI.

3.3 Lie symmetry based on FIEPL

The invariance of Eq.(25) under the infinitesimal transformation in Eq.(27) can obtian the following equations

$$\boldsymbol{X}^{(\beta)}\left\{\left(\frac{\partial R_\nu}{\partial a^\mu}{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)\cdot\right.$$
$$\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]+$$
$$\left.{}_\tau D_b^\beta\left[R_\mu\sin\left((\alpha-1)(t-\tau)+\frac{\pi}{2}\right)\right]\right\}=0$$
$$\mu=1,2,\cdots,2n \tag{35}$$

By substituting operator Eq.(30) into Eq.(35), we get

$$\boldsymbol{X}^{(0)}\left(\frac{\partial R_\nu}{\partial a^\mu}\right){}_aD_\tau^\beta a^\nu\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]-$$
$$\boldsymbol{X}^{(0)}\left(\frac{\partial B}{\partial a^\mu}\right)\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]+$$
$$\boldsymbol{X}^{(\beta)}\left\{{}_\tau D_b^\beta\left[R_\mu\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]\right]\right\}+$$
$$\xi_0(1-\alpha)\cos\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]\cdot$$
$$\left(\frac{\partial R_\nu}{\partial a^\mu}{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)+({}_aD_\tau^\beta\xi_\nu-{}_aD_\tau^\beta(\dot a^\nu\xi_0)+$$
$$\xi_0\,{}_aD_\tau^\beta\dot a^\nu)\frac{\partial R_\nu}{\partial a^\mu}\cdot\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]=0$$
$$\mu=1,2,\cdots,2n \tag{36}$$

Eq.(36) is the determining equation.

Definition 3 For the transformation Eq.(27), if the determining equations in Eq.(36) are satisfied, the symmetry is the Lie symmetry of fractional Birkhoffian system in Eq.(25) based on FIEPL.

4 Lie Symmetry Theorem of Fractional Birkhoffian System

Lie symmetry does not necessarily lead to conserved quantities. For the systems based on quasi-fractional models, the following theorems give the conditions and the conserved quantities led by Lie symmetry of this system.

4.1 Lie symmetry theorem based on RLFI

Theorem 1 If the generators ξ_0, ξ_μ satisfy the formula in Eq.(32) and there exists a gauge function $G=G(\tau,a^\nu)$ that satisfies the following structural equation

$$\left(\frac{\partial R_\nu}{\partial a^\mu}{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)\xi_\mu+\left(\frac{\partial R_\nu}{\partial \tau}{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial \tau}\right)\xi_0+$$
$$(R_\nu\,{}_aD_\tau^\beta a^\nu-B)\dot\xi_0+R_\nu\xi_0\,{}_aD_\tau^\beta\dot a^\nu+R_\nu\,{}_aD_\tau^\beta(\xi_\nu-$$
$$\dot a^\nu\xi_0)+(R_\nu\,{}_aD_\tau^\beta a^\nu-B)\xi_0\frac{1-\alpha}{t-\tau}=$$
$$-\dot G(t-\tau)^{1-\alpha} \tag{37}$$

then the fractional Birkhoffian system in Eq.(13) based on RLFI has the following conserved quantity

$$I=(R_\nu\,{}_aD_\tau^\beta a^\nu-B)\xi_0(t-\tau)^{\alpha-1}+$$
$$\int_a^\tau[R_\nu\,{}_aD_s^\beta(\xi_\nu-\dot a^\nu\xi_0)(t-s)^{\alpha-1}-$$
$${}_sD_b^\beta(R_\mu(t-s)^{\alpha-1})(\xi_\mu-\dot a^\mu\xi_0)]\,\mathrm{d}s+G=\text{const} \tag{38}$$

Proof

$$\frac{\mathrm{d}I}{\mathrm{d}\tau}=(R_\nu\,{}_aD_\tau^\beta a^\nu-B)\dot\xi_0(t-\tau)^{\alpha-1}+$$
$$\frac{\mathrm{d}(R_\nu\,{}_aD_\tau^\beta a^\nu-B)}{\mathrm{d}\tau}\xi_0(t-\tau)^{\alpha-1}+(R_\nu\,{}_aD_\tau^\beta a^\nu-$$
$$B)\xi_0(1-\alpha)(t-\tau)^{\alpha-2}+R_\nu\,{}_aD_\tau^\beta(\xi_\nu-$$
$$\dot a^\nu\xi_0)(t-\tau)^{\alpha-1}-{}_\tau D_b^\beta(R_\mu(t-\tau)^{\alpha-1})(\xi_\mu-$$
$$\dot a^\mu\xi_0)+\dot G \tag{39}$$

Substituting Eq.(37) into Eq.(39), we get

$$\frac{\mathrm{d}I}{\mathrm{d}\tau}=\frac{\mathrm{d}(R_\nu\,{}_aD_\tau^\beta a^\nu-B)}{\mathrm{d}\tau}\xi_0(t-\tau)^{\alpha-1}+R_\nu\,{}_aD_\tau^\beta(\xi_\nu-\dot{a}^\nu\xi_0)(t-\tau)^{\alpha-1}-{}_\tau D_b^\beta(R_\mu(t-\tau)^{\alpha-1})(\xi_\mu-\dot{a}^\mu\xi_0)-\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)\xi_\mu(t-\tau)^{\alpha-1}-\left(\frac{\partial R_\nu}{\partial\tau}\,{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial\tau}\right)\xi_0(t-\tau)^{\alpha-1}-R_\nu[{}_aD_\tau^\beta\dot{a}^\nu\xi_0+{}_aD_\tau^\beta(\xi_\mu-\dot{a}^\mu\xi_0)](t-\tau)^{\alpha-1}=\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\mu\,{}_aD_\tau^\beta a^\nu\xi_0(t-\tau)^{\alpha-1}-\frac{\partial B}{\partial a^\mu}\dot{a}^\mu\xi_0(t-\tau)^{\alpha-1}-{}_\tau D_b^\beta(R_\mu(t-\tau)^{\alpha-1})(\xi_\mu-\dot{a}^\mu\xi_0)-\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)\xi_\mu(t-\tau)^{\alpha-1}=-\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)(\xi_\mu-\dot{a}^\mu\xi_0)\cdot(t-\tau)^{\alpha-1}-(\xi_\mu-\dot{a}^\mu\xi_0)\,{}_\tau D_b^\beta(R_\mu(t-\tau)^{\alpha-1})$$

Due to Eq.(13), we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}\tau}=0$$

Therefore, the theorem is proved.

We can describe Theorem 1 as Lie symmetry theorem of system (13).

When $\beta\to1$, the fractional Birkhoff's equations in Eq.(13) are reduced to

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial\tau}=\frac{1-\alpha}{t-\tau}R_\mu \quad \mu=1,2,\cdots,2n \tag{40}$$

Eqs.(40) are Birkhoff's equations based on quasi-fractional model given in Ref.[56]. And Theorem 1 is reduced to the following theorem.

Theorem 2 If the generators ξ_0 and ξ_μ satisfy the following Lie symmetry equations

$$\boldsymbol{X}^{(0)}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu(t-\tau)^{\alpha-1}-\boldsymbol{X}^{(0)}\left(\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial\tau}\right)(t-\tau)^{\alpha-1}-\boldsymbol{X}^{(0)}[(1-\alpha)(t-\tau)^{\alpha-2}R_\mu]+\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)(\dot{\xi}_\nu-\dot{a}^\nu\dot{\xi}_0)(t-\tau)^{\alpha-1}+\xi_0(1-\alpha)(t-\tau)^{\alpha-2}\left[\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial\tau}\right]=0 \quad \mu=1,2,\cdots,2n \tag{41}$$

and there exists a gauge function $G=G(\tau,a^\nu)$ which satisfies the following structural equation

$$\left(\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}\right)\xi_\mu+\left(\frac{\partial R_\nu}{\partial\tau}\dot{a}^\nu-\frac{\partial B}{\partial\tau}\right)\xi_0+(R_\nu\dot{a}^\nu-B)\dot{\xi}_0+R_\nu(\dot{\xi}_\nu-\dot{a}^\nu\xi_0)+(R_\nu\dot{a}^\nu-B)\xi_0\frac{1-\alpha}{t-\tau}=-\dot{G}(t-\tau)^{1-\alpha} \tag{42}$$

then the system (40) has the following conserved quantity

$$I=(R_\nu\xi_\nu-B\xi_0)(t-\tau)^{\alpha-1}+G=\mathrm{const} \tag{43}$$

When $\beta\to1$ and $\alpha\to1$, Theorem 1 becomes the Lie symmetry theorem of classical Birkhoffian system[32].

4.2 Lie symmetry theorem based on EEFI

Theorem 3 If the generators ξ_0, ξ_μ satisfy Eq.(34) and there exists a gauge function $G=G(\tau,a^\nu)$ that satisfies the following structural equation

$$\left(\frac{\partial R_\nu}{\partial a^\mu}\,{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)\xi_\mu+\left(\frac{\partial R_\nu}{\partial\tau}\,{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial\tau}\right)\xi_0+(R_\nu\,{}_aD_\tau^\beta a^\nu-B)\dot{\xi}_0+R_\nu\xi_0\,{}_aD_\tau^\beta\dot{a}^\nu+R_\nu\,{}_aD_\tau^\beta(\xi_\mu-\dot{a}^\mu\xi_0)+(R_\nu\,{}_aD_\tau^\beta a^\nu-B)\xi_0\frac{(\alpha-1)\sinh\tau}{\cosh t-\cosh\tau}=-\dot{G}(\cosh t-\cosh\tau)^{1-\alpha} \tag{44}$$

then the fractional Birkhoffian system (19) based on EEFI has the following conserved quantity

$$I=(R_\nu\,{}_aD_\tau^\beta a^\nu-B)\xi_0(\cosh t-\cosh\tau)^{\alpha-1}+\int_a^\tau[R_\nu\,{}_aD_s^\beta(\xi_\nu-\dot{a}^\nu\xi_0)(\cosh t-\cosh\tau)^{\alpha-1}-{}_sD_b^\beta(R_\mu(\cosh t-\cosh\tau)^{\alpha-1})(\xi_\mu-\dot{a}^\mu\xi_0)]\mathrm{d}s+G=\mathrm{const} \tag{45}$$

Theorem 3 Theorem 3 can be called Lie symmetry theorem of fractional Birkhoffian system in Eq.(19) based on EEFI.

When $\beta\to1$, the fractional Birkhoff's equations in Eq.(19) are reduced to

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial\tau}=\frac{(\alpha-1)\sinh\tau}{\cosh t-\cosh\tau}R_\mu \quad \mu=1,2,\cdots,2n \tag{46}$$

Eqs.(46) are Birkhoff's equations based on quasi-fractional models given in Ref.[56]. And Theorem 3 is reduced to the following theorem.

Theorem 4 If the generators ξ_0,ξ_μ satisfy the following equations

$$\boldsymbol{X}^{(0)}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu(\cosh t-\cosh\tau)^{\alpha-1}-$$
$$\boldsymbol{X}^{(0)}\left(\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial\tau}\right)(\cosh t-\cosh\tau)^{\alpha-1}-$$
$$\boldsymbol{X}^{(0)}[(\alpha-1)\sinh\tau(\cosh t-\cosh\tau)^{\alpha-2}R_\mu]+$$
$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)(\dot{\xi}_\nu-$$
$$\dot{a}^\nu\dot{\xi}_0)(\cosh t-\cosh\tau)^{\alpha-1}+$$
$$\xi_0(\alpha-1)\sinh\tau(\cosh t-\cosh\tau)^{\alpha-2}\cdot$$
$$\left[\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial\tau}\right]=0 \tag{47}$$

and there exsits a gauge function $G=G(\tau,a^\nu)$ which satisfies the following structural equation

$$\left(\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}\right)\xi_\mu+\left(\frac{\partial R_\nu}{\partial\tau}\dot{a}^\nu-\frac{\partial B}{\partial\tau}\right)\xi_0+(R_\nu\dot{a}^\nu-$$
$$B)\dot{\xi}_0+R_\nu(\dot{\xi}_\mu-\dot{a}^\nu\xi_0)+(R_\nu\dot{a}^\nu-$$
$$B)\xi_0\frac{(\alpha-1)\sinh\tau}{\cosh t-\cosh\tau}=$$
$$-\dot{G}(\cosh t-\cosh\tau)^{1-\alpha} \tag{48}$$

then the system (46) contains the following conserved quantity

$$I=(R_\nu\xi_\nu-B\xi_0)(\cosh t-\cosh\tau)^{\alpha-1}+G=\text{const} \tag{49}$$

When $\beta\to1$ and $\alpha\to1$, Theorem 3 becomes Lie symmetry theorem of classical Birkhoffian system[32].

4.3 Lie symmetry theorem based on FIEPL

Theorem 5 If the generators ξ_0, ξ_μ satisfy Eq.(36) and there exists a gauge function $G=G(\tau,a^\nu)$ that satisfies the following structural equation

$$\left(\frac{\partial R_\nu}{\partial a^\mu}{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial a^\mu}\right)\xi_\mu+\left(\frac{\partial R_\nu}{\partial\tau}{}_aD_\tau^\beta a^\nu-\frac{\partial B}{\partial\tau}\right)\xi_0+$$
$$(R_\nu\,{}_aD_\tau^\beta a^\nu-B)\dot{\xi}_0+R_\nu\xi_0\,{}_aD_\tau^\beta\dot{a}^\nu+R_\nu\,{}_aD_\tau^\beta(\xi_\nu-$$
$$\dot{a}^\nu\xi_0)+(R_\nu\,{}_aD_\tau^\beta a^\nu-B)\xi_0(1-$$
$$\alpha)\cot\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]=$$
$$-\dot{G}\sin^{-1}\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right] \tag{50}$$

then the fractional Birkhoffian system in Eq.(25) based on FIEPL has the following conserved quantity

$$I=(R_\nu\,{}_aD_\tau^\beta a^\nu-B)\xi_0\cdot\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]+$$
$$\int_a^\tau\Big\{R_\nu\,{}_aD_s^\beta(\xi_\nu-\dot{a}^\nu\xi_0)\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]-$$
$${}_sD_b^\beta\left[R_\mu\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]\right](\xi_\mu-\dot{a}^\mu\xi_0)\Big\}\mathrm{d}s+$$
$$G=\text{const} \tag{51}$$

Theorem 5 Theorem 5 can be called Lie symmetry theorem of fractional Birkhoffian system in Eq.(25) based on FIEPL.

When $\beta\to1$, the fractional Birkhoff's equations in Eq.(25) become

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial\tau}=$$
$$R_\mu(1-\alpha)\cot\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]$$
$$\mu=1,2,\cdots,2n \tag{52}$$

Eqs.(52) are Birkhoff's equations based on quasi-fractional models given in Ref.[56]. And Theorem 5 is reduced to the following theorem

Theorem 6 If the infinitesimal generators ξ_0, ξ_μ satisfy the following equations

$$\boldsymbol{X}^{(0)}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\cdot\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]-$$
$$\boldsymbol{X}^{(0)}\left(\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial\tau}\right)\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]-$$
$$\boldsymbol{X}^{(0)}\left[(1-\alpha)\cos\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]R_\mu\right]+$$
$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)(\dot{\xi}_\nu-\dot{a}^\nu\dot{\xi}_0)\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]+$$
$$\xi_0(1-\alpha)\cos\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]\cdot$$
$$\left[\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial\tau}\right]=0$$
$$\mu=1,2,\cdots,2n \tag{53}$$

and there exists a gauge function $G=G(\tau,a^\nu)$ which satisfies the following structural equation

$$\left(\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}\right)\xi_\mu+\left(\frac{\partial R_\nu}{\partial\tau}\dot{a}^\nu-\frac{\partial B}{\partial\tau}\right)\xi_0+(R_\nu\dot{a}^\nu-$$
$$B)\dot{\xi}_0+R_\nu(\dot{\xi}_\mu-\dot{a}^\nu\xi_0)+(R_\nu\dot{a}^\nu-B)\xi_0(1-$$
$$\alpha)\cot\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]=$$
$$-\dot{G}\arcsin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right] \tag{54}$$

then the system (42) contains the following conserved quantity

$$I=(R_{\nu}\xi_{\mu}-B\xi_0)\sin\left[(\alpha-1)(t-\tau)+\frac{\pi}{2}\right]+G=\text{const} \tag{55}$$

When $\beta\rightarrow 1$ and $\alpha\rightarrow 1$, Theorem 5 becomes Lie symmetry theorem of classical Birkhoffian system[32].

5 Examples

(1) Example 1

A fractional Birkhoffian system is studied based on RLFI. Its Pfaff action is

$$S_{\mathrm{R}}=\frac{1}{\Gamma(\alpha)}\int_a^b[a^2\,{}_aD_\tau^\beta a^1+a^4\,{}_aD_\tau^\beta a^3-a^2a^3](t-\tau)^{\alpha-1}\mathrm{d}\tau \tag{56}$$

According to Eq.(13), Birkhoff's equations of the system can be written as

$$\begin{gathered}{}_\tau D_b^\beta[a^2(t-\tau)^{\alpha-1}]=0,\ {}_aD_\tau^\beta a^1-a^3=0\\ -a^2(t-\tau)^{\alpha-1}+{}_\tau D_b^\beta[a^4(t-\tau)^{\alpha-1}]=0\\ {}_aD_\tau^\beta a^3=0\end{gathered} \tag{57}$$

The structural Eq.(37) gives

$$\begin{aligned}&({}_aD_\tau^\beta a^1-a^3)\xi_2-a^2\xi_3+{}_aD_\tau^\beta a^3\xi_4+(a^2\,{}_aD_\tau^\beta a^1+a^4\,{}_aD_\tau^\beta a^3-a^2a^3)\dot{\xi}_0+a^2\xi_0\,{}_aD_\tau^\beta\dot{a}^1+\\&a^4\xi_0\,{}_aD_\tau^\beta\dot{a}^3+a^2\,{}_aD_\tau^\beta(\xi_1-\dot{a}^1\xi_0)+a^4\,{}_aD_\tau^\beta(\xi_3-\dot{a}^3\xi_0)+(a^2\,{}_aD_\tau^\beta a^1+a^4\,{}_aD_\tau^\beta a^3-a^2a^3)\xi_0\frac{1-\alpha}{t-\tau}=\\&-\dot{G}(t-\tau)^{1-\alpha}\end{aligned} \tag{58}$$

Then

$$\xi_0=1,\ \xi_1=a^1,\ \xi_2=\frac{\alpha-1}{t-\tau}a^2,\ \xi_3=a^3,\ \xi_4=0 \tag{59}$$

The generators in Eq.(59) satisfy Eq.(32). Substituting the generators in Eq.(59) into Eq.(58), we have

$$\dot{G}=0 \tag{60}$$

According to Theorem 1, we obtain

$$\begin{aligned}I=&(a^2\,{}_aD_\tau^\beta a^1+a^4\,{}_aD_\tau^\beta a^3-a^2a^3)(t-\tau)^{\alpha-1}+\int_a^\tau\Big\{a^2\,{}_aD_s^\beta(a^1-\dot{a}^1)(t-s)^{\alpha-1}+\\&a^4\,{}_aD_s^\beta(a^3-\dot{a}^3)(t-s)^{\alpha-1}-(a^1-\dot{a}^1)\,{}_sD_b^\beta[a^2(t-s)^{\alpha-1}]-\\&(a^3-\dot{a}^3)\,{}_sD_b^\beta[a^4(t-s)^{\alpha-1}]\Big\}\mathrm{d}s\end{aligned} \tag{61}$$

Eq.(61) is the conserved quantity.

When $\beta\rightarrow 1$, Eq.(61) is reduced to

$$I=(a^2a^1+a^4a^3-a^2a^3)(t-\tau)^{\alpha-1} \tag{62}$$

Eq.(62) is the conserved quantity of the quasi-fractional Birkhoffian system.

When $\beta\rightarrow 1,\alpha\rightarrow 1$, Eq.(61) is reduced to

$$I=a^2a^1+a^4a^3-a^2a^3 \tag{63}$$

Eq.(63) is the conserved quantity of classical Birkhoffian system.

(2) Example 2

A fractional Birkhoffian system based on EEFI is elabrated. Its Pfaff action is

$$S_{\mathrm{E}}=\frac{1}{\Gamma(\alpha)}\int_a^b\left[a^3\,{}_aD_\tau^\beta a^1+a^4\,{}_aD_\tau^\beta a^2-\left(\frac{1}{2}(a^3)^2+\frac{1}{2}(a^4)^2\right)\right](\cosh t-\cosh\tau)^{\alpha-1}\mathrm{d}\tau \tag{64}$$

Birkhoff's equations based on EEFI are

$$\begin{gathered}{}_\tau D_b^\beta[a^3(\cosh t-\cosh\tau)^{\alpha-1}]=0\\ {}_\tau D_b^\beta[a^4(\cosh t-\cosh\tau)^{\alpha-1}]=0\\ {}_aD_\tau^\beta a^1-a^3=0,\ {}_aD_\tau^\beta a^2-a^4=0\end{gathered} \tag{65}$$

The structural Eq.(44) gives

$$\begin{aligned}&({}_aD_\tau^\beta a^1-a^3)\xi_3+({}_aD_\tau^\beta a^2-a^4)\xi_4+\left\{a^3\,{}_aD_\tau^\beta a^1+a^4\,{}_aD_\tau^\beta a^2-\left[\frac{1}{2}(a^3)^2+\frac{1}{2}(a^4)^2\right]\right\}\dot{\xi}_0+a^3\xi_0\,{}_aD_\tau^\beta\dot{a}^1+\\&a^4\xi_0\,{}_aD_\tau^\beta\dot{a}^2+a^3\,{}_aD_\tau^\beta(\xi_1-\dot{a}^1\xi_0)+a^4\,{}_aD_\tau^\beta(\xi_2-\dot{a}^2\xi_0)+\left\{a^3\,{}_aD_\tau^\beta a^1+a^4\,{}_aD_\tau^\beta a^2-\right.\\&\left.\left[\frac{1}{2}(a^4)^2+\frac{1}{2}(a^3)^2\right]\right\}\xi_0\frac{(\alpha-1)\sinh\tau}{\cosh t-\cosh\tau}=-\dot{G}(\cosh t-\cosh\tau)^{1-\alpha}\end{aligned} \tag{66}$$

Then

$$\xi_0=0,\ \xi_1=a^2,\ \xi_2=-a^1,\ \xi_3=0,\ \xi_4=0 \tag{67}$$

The generators in Eq.(67) satisfy Eq.(34). Substituting the generators in Eq.(67) into Eq.(66), we have

$$\dot{G}=0 \tag{68}$$

According to Theorem 3, we obtain

$$\begin{aligned}I=\int_a^\tau\Big\{&a^3\,{}_aD_s^\beta a^2(\cosh t-\cosh s)^{\alpha-1}-a^4\,{}_aD_s^\beta a^1(\cosh t-\cosh s)^{\alpha-1}-\\&a^2\,{}_sD_b^\beta[a^3(\cosh t-\cosh s)^{\alpha-1}]+a^1\,{}_sD_b^\beta[a^4(\cosh t-\cosh s)^{\alpha-1}]\Big\}\mathrm{d}s\end{aligned} \tag{69}$$

Eq.(69) is the conserved quantity.

When $\beta \to 1$, from Eq.(69), we have

$$I = (a^3 a^2 - a^4 a^1)(\cosh t - \cosh \tau)^{\alpha - 1} \tag{70}$$

Eq.(70) is the conserved quantity of Birkhoffian system based on EEFI.

When $\beta \to 1, \alpha \to 1$, Eq.(69) becomes

$$I = a^3 a^2 - a^4 a^1 \tag{71}$$

Eq.(71) is the conserved quantity of classical Birkhoffian system.

(3) Example 3

Now we study a fractional Birkhoffian system based on FIEPL. Its Pfaff action is

$$S_{\mathrm{P}} = \frac{1}{\Gamma(\alpha)} \int_a^b [(a^2 + a^3)\, {}_aD_\tau^\beta a^1 + a^4\, {}_aD_\tau^\beta a^3 - (a^2 a^3 + (a^3)^2)] \sin\left[(\alpha - 1)(t - \tau) + \frac{\pi}{2}\right] \mathrm{d}\tau \tag{72}$$

Birkhoff's equations under FIEPL are

$$\begin{gathered} {}_\tau D_b^\beta \left[(a^2 + a^3) \sin\left((\alpha - 1)(t - \tau) + \frac{\pi}{2}\right)\right] = 0 \\ \left({}_aD_\tau^\beta a^1 - a^2 - 2a^3\right) \sin\left[(\alpha - 1)(t - \tau) + \frac{\pi}{2}\right] + {}_\tau D_b^\beta \left[a^4 \sin\left[(\alpha - 1)(t - \tau) + \frac{\pi}{2}\right]\right] = 0 \\ {}_aD_\tau^\beta a^1 - a^3 = 0, {}_aD_\tau^\beta a^3 = 0 \end{gathered} \tag{73}$$

The structural Eq.(50) gives

$$\begin{gathered} ({}_aD_\tau^\beta a^1 - a^3) \xi_2 + ({}_aD_\tau^\beta a^1 - a^2 - 2a^3) \xi_3 + {}_aD_\tau^\beta a^3 \xi_4 + ((a^2 + a^3)\, {}_aD_\tau^\beta a^1 + a^4\, {}_aD_\tau^\beta a^3 - a^2 a^3 - (a^3)^2) \dot{\xi}_0 + (a^2 + a^3) \xi_0\, {}_aD_\tau^\beta \dot{a}^1 + a^4 \xi_0\, {}_aD_\tau^\beta \dot{a}^3 + \\ (a^2 + a^3)\, {}_aD_\tau^\beta (\xi_1 - \dot{a}^1 \xi_0) + a^4\, {}_aD_\tau^\beta (\xi_3 - \dot{a}^3 \xi_0) + ((a^2 + a^3)\, {}_aD_\tau^\beta a^1 + a^4\, {}_aD_\tau^\beta a^3 - a^2 a^3 - \\ (a^3)^2) \xi_0 (1 - \alpha) \cdot \cot\left[(\alpha - 1)(t - \tau) + \frac{\pi}{2}\right] = -\dot{G} \operatorname{arcsin}\left[(\alpha - 1)(t - \tau) + \frac{\pi}{2}\right] \end{gathered} \tag{74}$$

It is easy to verify that the system has the Lie symmetry

$$\xi_0 = 1,\ \xi_1 = a^1,\ \xi_2 = 0,\ \xi_3 = a^3,\ \xi_4 = 0 \tag{75}$$

According to Theorem 5, corresponding to Lie symmetry in Eq.(75), the system has the following conserved quantity

$$\begin{aligned} I = {} & ((a^2 + a^3)\, {}_aD_\tau^\beta a^1 + a^4\, {}_aD_\tau^\beta a^3 - a^2 a^3 - (a^3)^2) \sin\left[(\alpha - 1)(t - \tau) + \frac{\pi}{2}\right] + \\ & \int_a^\tau \Big\{ (a^2 + a^3)\, {}_aD_s^\beta (a^1 - \dot{a}^1) \cdot \sin\left[(\alpha - 1)(t - s) + \frac{\pi}{2}\right] + a^4\, {}_aD_s^\beta (a^3 - \dot{a}^3) \sin\left[(\alpha - 1)(t - s) + \frac{\pi}{2}\right] - (a^1 - \\ & \dot{a}^1) \cdot {}_sD_b^\beta \left[(a^2 + a^3) \sin\left((\alpha - 1)(t - s) + \frac{\pi}{2}\right)\right] - (a^3 - \dot{a}^3)\, {}_sD_b^\beta \left[a^4 \sin\left((\alpha - 1)(t - s) + \frac{\pi}{2}\right)\right] \Big\} \mathrm{d}s \end{aligned} \tag{76}$$

If $\beta \to 1$, we get

$$I = [(a^2 + a^3) a^1 + a^4 a^3 - a^2 a^3 - (a^3)^2] \sin\left[(\alpha - 1)(t - \tau) + \frac{\pi}{2}\right] \tag{77}$$

Eq.(76) is the conserved quantity of Birkhoffian system based on FIEPL.

If $\beta \to 1$, and $\alpha \to 1$, we get

$$I = (a^2 + a^3) a^1 + a^4 a^3 - a^2 a^3 - (a^3)^2 \tag{78}$$

Eq.(78) is the conserved quantity of classical Birkhoffian system.

6 Conclusions

Since the fractional models can describe the dynamics behavior of complex systems more accurately, the research of fractional Birkhoffian system has always been a hot topic. The quasi-fractional dynamics models can simplify some complex problems in non-conservative dynamics, so it is quite valuable to study the symmetry of fractional Birkhoffian system based on quasi-fractional dynamics models. The Lie symmetry of fractional Birkhoffian system based on quasi-fractional dynamics models is proposed and studied in this paper. The determining equations for the systems based on RLFI, EEFI and FIEPL are established seperately. The corresponding structural Eqs.(44, 50) and the conserved quantities of Eqs.(39, 45, 51) are obtained. The Lie symmetry theorems of quasi-fractional Birkhoffian systems and classical Birkhoffian systems are special cases of this paper.

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基于准分数阶模型的分数阶Birkhoff动力学及其Lie对称性

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摘要:为了探究非保守系统的动力学行为,该文提出并研究基于准分数阶动力学模型的分数阶Birkhoff动力学的Lie对称性和守恒量。准分数阶动力学模型是指基于Riemann-Liouville分数阶积分定义的变分问题、基于按指数律扩展的分数阶积分定义的变分问题和基于按周期函数律拓展的分数阶积分定义的变分问题。首先,建立了基于准分数阶模型的分数阶Pfaff-Birkhoff原理,得到了相应的Birkhoff方程和Lie对称性确定方程。其次,对于基于准分数阶模型的分数阶Birkhoff系统,给出了守恒量的条件和形式,并证明了Lie对称性定理。准分数阶Birkhoff系统与经典Birkhoff系统的Pfaff-Birkhoff原理、Birkhoff方程和Lie对称定理均是该文的特例。最后,给出了若干算例。

关键词:准分数阶动力学模型;Lie对称性;守恒量;分数阶Birkhoff系统;Riemann-Liouville导数