Analysis of Vibration Frequencies of Piezoelectric Ceramic Rings as Ultrasonic Transducers in Welding of Facial Mask Production

WU Jinghui, *ZHAO Shengquan*, *XIE Longtao*, *WANG Ji**

Piezoelectric Device Laboratory, School of Mechanical Engineering & Mechanics, Ningbo University, Ningbo 315211, P. R. China

(Received 2 November 2020; revised 20 January 2021; accepted 22 January 2021)

Abstract: The explosive demands for facial masks as vital personal protection equipment (PPE) in the wake of Covid-19 have challenged many industries and enterprises in technology and capacity, and the piezoelectric ceramic (PZT) transducers for the production of facial masks in the welding process are in heavy demand. In the earlier days of the epidemic, the supply of ceramic transducers cannot meet its increasing demands, and efforts in materials, development, and production are mobilized to provide the transducers to mask producers for quick production. The simplest solution is presented with the employment of Rayleigh-Ritz method for the vibration analysis, then different materials can be selected to achieve the required frequency and energy standards. The fully tailored method and results can be utilized by the engineers for quick development of the PZT transducers to perform precise function in welding.

Key words: piezoelectric ceramic (PZT); transducer; facial mask welding; personal protection equipment (PPE)

CLC number: TB12 **Document code:** A **Article ID:** 1005-1120(2021)01-0124-08

0 Introduction

As one of the important personal protection equipment (PPE), facial masks of various specifications are widely demanded in a sudden due to explosive and simultaneous outbreaks of Covid-19 worldwide. Since facial mask is one of the major regulations widely accepted and practiced around the world, people are suggested or even required mandatorily to wear facial masks in highly risky environments. Therefore, facial mask demands surge worldwide. Many mask production lines are built and operating in a hurry yet beyond the capacity. All components needed in the production lines are in short supply. Among them, piezoelectric ceramic (PZT) ring for the ultrasonic welding process is one of the highly sought components related to the mask production with limited supply. When the epidemic spreads, many efforts have been made to produce the transducers quickly or to develop replacements. In the design process before production, it is critical to analyze the essential properties of ceramic ring in assistance with proper tools and procedure, such as vibration frequency and mode. As the obligation to China and global community, authors have released a guide on specifications, materials, and design considerations of PZT rings for facial mask production lines. PZT ring is used to generate ultrasonic waves of specific frequency, thus enabling the welding of mask fibers at a frequency of 15 kHz and 20 kHz with a piezoelectric cylinder dimension of approximately 50 mm×17 mm ×7 mm[1]. The subsequent development focuses on the efficient and simple procedures for accurate analysis of vibrations of ceramic

*Corresponding author, E-mail address: wangji@nbu.edu.cn.

How to cite this article: WU Jinghui, ZHAO Shengquan, XIE Longtao, et al. Analysis of vibration frequencies of piezoelectric ceramic rings as ultrasonic transducers in welding of facial mask production[J]. Transactions of Nanjing University of Aeronautics and Astronautics, 2021, 38(1): 124-131.

http://dx.doi.org/10.16356/j.1005-1120.2021.01.012

rings presented in this paper.

The ceramic ring is usually made of PZT ceramic with large elastic constants, or the hard PZT, for the generation of appropriate ultrasonic waves needed in the welding process. Most transducers for this objective are made of PZT-4, but as a replacement of PZT-4, PZT-8 can also have equivalent properties. For the high frequency vibrations of the PZT ring, the analysis is not a straightforward task with readily available methods and results, due to the complicated nature of finite elastic solids with material anisotropy[2-3]. The analysis should be performed by either approximate or numerical methods, which are widely available but not specific to the particular transducer, and the process can be a challenge to many engineers. Therefore, recapitulated the common procedure of the analysis with the Rayleigh-Ritz method, a detailed formulation is provided so that engineers can analyze the parameters of PZT rings with a reliable guide in a fast manner[4]. In addition, results of both PZT-4 and -8 are demonstrated to give examples for material choices. All procedure and codes of the program are available on the website of Piezoelectric Device Laboratory (piezo.nbu.edu.cn). We promise to provide full support on the analysis with the method outlined in this paper.

The formulation here is straightforward with the theory of piezoelectricity and wave propagation, which can be found in Refs.[5-8]. Although the differential equations of the vibration of a piezoelectric ring are well-known and the general procedure of analytical solutions have been discussed[5,9-10], the closed form solutions are not available. The most effective method for the solution is the Rayleigh-Ritz method[11-12]. In this paper, the Rayleigh-Ritz method is tailored for this transducer with improved accuracy and simplicity.

1 Mathematical Model

A typical PZT ring is shown in Fig. 1, where R_1, R_0 are radii and h is the height. The PZT is always polarized in the z direction in practice. The parameters shown in Fig.1 and the material constants will be used in the following analysis.

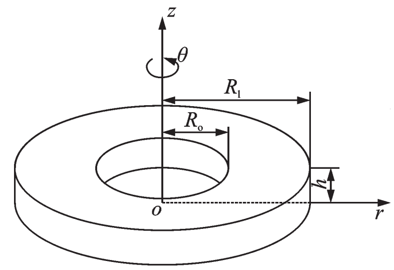

Fig.1 Typical PZT ring

Considering deformation and electric field, we define the generalized strain energy in cylindrical coordinate as

$$V=\int_0^{2\pi}\int_{R_0}^{R_1}\int_{-h/2}^{h/2}\left(\frac{1}{2}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{S}-\boldsymbol{S}^{\mathrm{T}}\boldsymbol{e}\boldsymbol{E}-\frac{1}{2}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{\varepsilon}\boldsymbol{E}\right)r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta \tag{1}$$

where $\boldsymbol{C}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{e}$ are the elastic, the piezoelectric, and the dielectric material constants. The components of the strain and the electric field are

$$\boldsymbol{S}=[S_{rr}\quad S_{\theta\theta}\quad S_{zz}\quad S_{\theta z}\quad S_{rz}\quad S_{r\theta}]^{\mathrm{T}}$$
$$\boldsymbol{E}=[E_r\quad E_\theta\quad E_z]^{\mathrm{T}} \tag{2}$$

For a typical PZT material, the material matrices of the piezoelectric, the elastic, and the dielectric constants are

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_{11}&0&0\\0&\varepsilon_{11}&0\\0&0&\varepsilon_{33}\end{bmatrix}$$

$$\boldsymbol{C}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&0&0&0\\C_{12}&C_{11}&C_{13}&0&0&0\\C_{13}&C_{13}&C_{33}&0&0&0\\0&0&0&C_{44}&0&0\\0&0&0&0&C_{44}&0\\0&0&0&0&0&C_{66}\end{bmatrix}$$

$$\boldsymbol{e}=\begin{bmatrix}0&0&e_{31}\\0&0&e_{31}\\0&0&e_{33}\\0&e_{15}&0\\e_{15}&0&0\\0&0&0\end{bmatrix} \tag{3}$$

The element values of the above matrices will be given later for specific materials.

Naturally, the kinetic energy of the PZT ring is

$$T=\frac{\rho}{2}\int_0^{2\pi}\int_{R_0}^{R_1}\int_{-h/2}^{h/2}\left(\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right)r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta \tag{4}$$

where ρ is the density of ceramic materials; u, v, w are the displacements; and t is the time.

Given the displacements and the electric potential of the ceramic ring, components of strain and electric field as generalized strains in cylindrical coordinates are

$$\begin{aligned}
&S_{rr}=\frac{\partial u}{\partial r}, && S_{\theta\theta}=\frac{u}{r}+\frac{\partial v}{r\partial\theta}, && S_{zz}=\frac{\partial w}{\partial z}\\
&S_{r\theta}=\frac{\partial u}{r\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r}, && S_{rz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}, && S_{\theta z}=\frac{\partial v}{\partial z}+\frac{\partial w}{r\partial\theta}\\
&E_{rr}=-\frac{\partial\varphi}{\partial r}, && E_{\theta\theta}=-\frac{1}{r}\frac{\partial\varphi}{\partial\theta}, && E_{zz}=-\frac{\partial\varphi}{\partial z}
\end{aligned} \tag{5}$$

where φ is the electric potential.

For simplicity, the normalized coordinates are

$$\bar{r}=\frac{2r}{\bar{R}}-\delta, \bar{R}=R_1-R_0, \delta=\frac{R_1+R_0}{R_1-R_0}, \bar{\theta}=\theta, \bar{z}=\frac{2z}{h} \tag{6}$$

For free vibrations, the displacements and electric potential of the ceramic ring in normalized coordinates are

$$\begin{aligned}
u(r,\theta,z,t)&=U(\bar{r},\bar{\theta},\bar{z})\mathrm{e}^{\mathrm{i}\omega t}\\
v(r,\theta,z,t)&=V(\bar{r},\bar{\theta},\bar{z})\mathrm{e}^{\mathrm{i}\omega t}\\
w(r,\theta,z,t)&=W(\bar{r},\bar{\theta},\bar{z})\mathrm{e}^{\mathrm{i}\omega t}\\
\varphi(r,\theta,z,t)&=\Phi(\bar{r},\bar{\theta},\bar{z})\mathrm{e}^{\mathrm{i}\omega t}
\end{aligned} \tag{7}$$

where ω is the frequency of vibration and $\mathrm{i}=\sqrt{-1}$.

In cylindrical coordinates, functions of the amplitudes of displacements and electric potential of the ceramic ring can be further written as

$$\begin{aligned}
U(\bar{r},\bar{\theta},\bar{z})&=\bar{U}(\bar{r},\bar{z})\cos(n\bar{\theta})\\
V(\bar{r},\bar{\theta},\bar{z})&=\bar{V}(\bar{r},\bar{z})\sin(n\bar{\theta})\\
W(\bar{r},\bar{\theta},\bar{z})&=\bar{W}(\bar{r},\bar{z})\cos(n\bar{\theta})\\
\Phi(\bar{r},\bar{\theta},\bar{z})&=\bar{\Phi}(\bar{r},\bar{z})\cos(n\bar{\theta})
\end{aligned} \tag{8}$$

where n is an integer representing the circumferential wavenumber. If $n=0$, the vibration is axisymmetric.

Now the generalized strain energy in Eq.(1) with normalized coordinates is

$$\begin{aligned}
V_{\max}=&\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\Big\{\frac{\Gamma_1}{2}\Big[C_{11}(S_{\bar{r}\bar{r}}^2+S_{\bar{\theta}\bar{\theta}}^2)+\\
&2C_{12}\bar{S}_{\bar{r}\bar{r}}\bar{S}_{\bar{\theta}\bar{\theta}}+2C_{13}(\bar{S}_{\bar{r}\bar{r}}\bar{S}_{\bar{z}\bar{z}}+\bar{S}_{\bar{\theta}\bar{\theta}}\bar{S}_{\bar{z}\bar{z}})+\\
&C_{33}\bar{S}_{\bar{z}\bar{z}}^2+C_{44}\bar{S}_{\bar{r}\bar{z}}^2\Big]-\frac{1}{2}(\varepsilon_{11}\bar{E}_{\bar{r}\bar{r}}^2+\varepsilon_{33}\bar{E}_{\bar{z}\bar{z}}^2)+\\
&\frac{\Gamma_2}{2}(C_{44}\bar{S}_{\bar{\theta}\bar{z}}^2+C_{66}\bar{S}_{\bar{r}\bar{\theta}}^2)-\frac{1}{2}\varepsilon_{11}\bar{E}_{\bar{\theta}\bar{\theta}}^2-\\
&\Gamma_1\Big[e_{15}\bar{S}_{\bar{r}\bar{z}}\bar{E}_{rr}+(e_{31}\bar{S}_{\bar{r}\bar{r}}+e_{31}\bar{S}_{\bar{\theta}\bar{\theta}}+\\
&e_{33}\bar{S}_{\bar{z}\bar{z}})\bar{E}_{zz}\Big]-\Gamma_2 e_{15}\bar{S}_{\bar{\theta}\bar{z}}\bar{E}_{\bar{\theta}\bar{\theta}}\Big\}(\bar{r}+\delta)\mathrm{d}\bar{r}\mathrm{d}\bar{z}
\end{aligned} \tag{9}$$

The kinetic energy is

$$T_{\max}=\frac{\rho\bar{R}^2\omega^2h}{16}\int_{-1}^{1}\int_{-1}^{1}(\Gamma_1\bar{U}^2+\Gamma_2\bar{V}^2+\Gamma_1\bar{W}^2)(\bar{r}+\delta)\mathrm{d}\bar{r}\mathrm{d}\bar{z} \tag{10}$$

where the components of the generalized strain and parameters are

$$\begin{aligned}
&\bar{S}_{\bar{r}\bar{r}}=\frac{\partial\bar{U}}{\partial\bar{r}}, \bar{S}_{\bar{\theta}\bar{\theta}}=\frac{\bar{U}}{\bar{r}+\delta}+\frac{n\bar{V}}{\bar{r}+\delta}\\
&\bar{S}_{\bar{z}\bar{z}}=\frac{\partial\bar{W}}{\delta\partial\bar{z}}, \bar{S}_{\bar{\theta}\bar{z}}=\frac{\partial\bar{V}}{\delta\partial\bar{z}}-\frac{n\bar{W}}{\bar{r}+\delta}\\
&\bar{S}_{\bar{r}\bar{\theta}}=-\frac{n\bar{U}}{\bar{r}+\delta}+\frac{\partial\bar{V}}{\partial\bar{r}}-\frac{\bar{V}}{\bar{r}+\delta}\\
&\bar{S}_{\bar{r}\bar{z}}=\frac{\partial\bar{U}}{\gamma\partial\bar{z}}+\frac{\partial\bar{W}}{\partial\bar{r}}, \bar{E}_{\bar{r}\bar{r}}=-\frac{\partial\bar{\Phi}}{\partial\bar{r}}\\
&\bar{E}_{\bar{\theta}\bar{\theta}}=-\frac{n\bar{\Phi}}{\bar{r}+\delta}, \bar{E}_{\bar{z}\bar{z}}=-\frac{\partial\bar{\Phi}}{\gamma\partial\bar{z}}, \gamma=\frac{h}{\bar{R}}\\
&\Gamma_1=\int_0^{2\pi}\cos^2(n\bar{\theta})\mathrm{d}\bar{\theta}=\begin{cases}2\pi & n=0\\ \pi & n>0\end{cases}\\
&\Gamma_2=\int_0^{2\pi}\sin^2(n\bar{\theta})\mathrm{d}\bar{\theta}=\begin{cases}0 & n=0\\ \pi & n>0\end{cases}
\end{aligned} \tag{11}$$

The amplitudes of the mechanical displacements and electric potential are expressed as the product of Chebyshev polynomials and boundary functions.

$$\begin{aligned}
\bar{U}(\bar{r},\bar{z})&=F_u(\bar{z})\sum_{i=1}^{I}\sum_{j=1}^{J}A_{ij}P_i(\bar{r})P_j(\bar{z})\\
\bar{V}(\bar{r},\bar{z})&=F_v(\bar{z})\sum_{k=1}^{K}\sum_{l=1}^{L}B_{kl}P_k(\bar{r})P_l(\bar{z})\\
\bar{W}(\bar{r},\bar{z})&=F_w(\bar{z})\sum_{m=1}^{M}\sum_{n=1}^{N}C_{mn}P_m(\bar{r})P_n(\bar{z})\\
\bar{\Phi}(\bar{r},\bar{z})&=F_\varphi(\bar{z})\sum_{o=1}^{O}\sum_{p=1}^{P}D_{op}P_o(\bar{r})P_p(\bar{z})
\end{aligned} \tag{12}$$

where the boundary functions are products of the two functions satisfying the boundary conditions at the two ends ($\bar{z}=\mp 1$). In Eq. (12), all the displacements satisfy the cylindrical conditions of the ring, and the electric potential satisfies the boundary conditions of electric field of free vibrations by setting the boundary functions to 1. The integers I, J, K, L, M, N are the order of the Chebyshev polynomials for the displacements and potential evaluation. They take the same value in the actual computation. The coefficients $A_{ij}, B_{kl}, C_{mn}, D_{op}$ are the amplitudes to be determined. The sth-order Chebyshev polynomials $P_s(\chi)$ $(s=i,j,k,l,m,n,o,p;\ \chi=\bar{r},\bar{z})$ used here are

$$P_s(\chi)=\cos[(s-1)\arccos(\chi)]\quad s=1,2,3,\cdots \tag{13}$$

The energy functional of the PZT ring is

$$\Pi=V_{\max}-T_{\max} \tag{14}$$

With the Rayleigh-Ritz method, the derivatives are

$$\frac{\partial\Pi}{\partial A_{ij}}=0,\quad \frac{\partial\Pi}{\partial B_{kl}}=0,\quad \frac{\partial\Pi}{\partial C_{mn}}=0,\quad \frac{\partial\Pi}{\partial D_{op}}=0 \tag{15}$$

Hence, the eigenvalue for the free vibrations of the ceramic ring is

$$\left\{\begin{bmatrix} \boldsymbol{K}_{uu} & \boldsymbol{K}_{uv} & \boldsymbol{K}_{uw} & \boldsymbol{K}_{u\varphi} \\ \boldsymbol{K}_{uv}^{\mathrm{T}} & \boldsymbol{K}_{vv} & \boldsymbol{K}_{vw} & \boldsymbol{K}_{v\varphi} \\ \boldsymbol{K}_{uw}^{\mathrm{T}} & \boldsymbol{K}_{vw}^{\mathrm{T}} & \boldsymbol{K}_{ww} & \boldsymbol{K}_{w\varphi} \\ \boldsymbol{K}_{u\varphi}^{\mathrm{T}} & \boldsymbol{K}_{v\varphi}^{\mathrm{T}} & \boldsymbol{K}_{w\varphi}^{\mathrm{T}} & \boldsymbol{K}_{\varphi\varphi} \end{bmatrix} - \bar{\Omega}^2 \begin{bmatrix} \boldsymbol{M}_{uu} & 0 & 0 & 0 \\ 0 & \boldsymbol{M}_{vv} & 0 & 0 \\ 0 & 0 & \boldsymbol{M}_{ww} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\} \begin{Bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \\ \boldsymbol{D} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{16}$$

where $\bar{\Omega}=\bar{R}\omega\sqrt{\rho}$, $\boldsymbol{K}_{\alpha\beta}$ and $\boldsymbol{M}_{\alpha\beta}$ $(\alpha,\beta=u,v,w,\varphi)$ are submatrices of stiffness and mass of different fields, and $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$ the amplitude vectors

$$\begin{aligned} \boldsymbol{A}&=[A_{11}\cdots A_{1J}\ A_{21}\cdots A_{2J}\ \cdots A_{I1}\cdots A_{IJ}]^{\mathrm{T}} \\ \boldsymbol{B}&=[B_{11}\cdots B_{1L}\ B_{21}\cdots B_{2L}\ \cdots B_{K1}\cdots B_{KL}]^{\mathrm{T}} \\ \boldsymbol{C}&=[C_{11}\cdots C_{1N}\ C_{21}\cdots C_{2N}\ \cdots C_{M1}\cdots C_{MN}]^{\mathrm{T}} \\ \boldsymbol{D}&=[D_{11}\cdots D_{1P}\ D_{21}\cdots D_{2P}\ \cdots D_{o1}\cdots D_{OP}]^{\mathrm{T}} \end{aligned} \tag{17}$$

For the composition of stiffness and mass submatrices, the subscripts u, v, w, φ represent the displacements and electric potential; i, k, m, o the orders of the Chebyshev polynomial $P_s(\bar{r})$; and s, j, l, n, p the orders of Chebyshev polynomial $P_s(\bar{z})$. Consequently, the elements of stiffness and mass matrices are

$$\boldsymbol{K}_{uu}=\Gamma_1\Big[C_{11}(D^{111}_{uiu\bar{i}}+D^{00-1}_{uiu\bar{i}})H^{00}_{uju\dot{j}}+\frac{C_{44}}{\gamma^2}D^{001}_{uiu\bar{i}}H^{11}_{uju\dot{j}}+C_{12}(D^{010}_{uiu\bar{i}}+D^{100}_{uiu\bar{i}})\cdot H^{00}_{uju\dot{j}}\Big]+\Gamma_2C_{66}n^2D^{00-1}_{uiu\bar{i}}H^{00}_{uju\dot{j}}$$

$$\boldsymbol{K}_{uv}=\Gamma_1(C_{11}nD^{00-1}_{uiv\dot{k}}+C_{12}nD^{100}_{uiv\dot{k}})H^{00}_{ujv\dot{l}}+\Gamma_2C_{66}n(D^{00-1}_{uiv\bar{k}}-D^{010}_{uiv\bar{k}})H^{00}_{uju\dot{l}}$$

$$\boldsymbol{K}_{uw}=\Gamma_1\left[C_{13}\frac{1}{\gamma}(D^{101}_{uiw\dot{m}}+D^{000}_{uiw\dot{m}})H^{01}_{ujw\dot{n}}+C_{44}\frac{1}{\gamma}D^{011}_{uiw\dot{m}}H^{10}_{ujw\dot{n}}\right]$$

$$\boldsymbol{K}_{u\varphi}=\Gamma_1\left[e_{31}\frac{1}{\gamma}(D^{101}_{ui\varphi\dot{o}}+D^{000}_{ui\varphi\dot{o}})H^{01}_{uj\varphi\dot{p}}+e_{15}\frac{1}{\gamma}D^{011}_{ui\varphi\dot{o}}H^{10}_{uj\varphi\dot{p}}\right]$$

$$\boldsymbol{K}_{vv}=\Gamma_2\left[C_{66}(D^{111}_{vkv\dot{k}}+D^{00-1}_{vkv\dot{k}}-D^{100}_{vkv\dot{k}})H^{00}_{vlv\dot{l}}+C_{44}\frac{1}{\gamma^2}D^{001}_{vkv\dot{k}}H^{11}_{vlv\dot{l}}\right]+\Gamma_1C_{11}n^2D^{00-1}_{vkv\dot{k}}H^{00}_{vlv\dot{l}}$$

$$\boldsymbol{K}_{vw}=\Gamma_1C_{13}\frac{n}{\gamma}D^{001}_{vkw\dot{m}}H^{01}_{vlw\dot{n}}-\Gamma_2C_{44}\frac{n}{\gamma}D^{000}_{vkw\dot{m}}H^{10}_{vlw\dot{n}}$$

$$\boldsymbol{K}_{v\varphi}=\Gamma_1e_{31}\frac{n}{\gamma}D^{000}_{vk\varphi\dot{o}}H^{01}_{vl\varphi\dot{p}}-\Gamma_2e_{15}\frac{n}{\gamma}D^{000}_{vk\varphi\dot{o}}H^{10}_{vl\varphi\dot{p}}$$

$$\boldsymbol{K}_{ww}=\Gamma_1\Big(C_{33}\frac{1}{\gamma^2}D^{001}_{wmw\dot{m}}H^{11}_{wnw\dot{n}}+C_{44}D^{111}_{wmw\dot{m}}\cdot H^{00}_{wnw\dot{n}}\Big)+\Gamma_2C_{44}n^2D^{00-1}_{wmw\dot{m}}H^{00}_{wnw\dot{n}},$$

$$\boldsymbol{K}_{w\varphi}=\Gamma_1\Big(e_{15}D^{111}_{wm\varphi\dot{o}}H^{00}_{wnw\dot{p}}+e_{33}\frac{1}{\gamma^2}D^{001}_{wm\varphi\dot{o}}\cdot H^{11}_{wnw\dot{p}}\Big)+\Gamma_2e_{15}n^2D^{00-1}_{wm\varphi\dot{o}}H^{00}_{wnw\dot{p}},$$

$$\boldsymbol{K}_{\varphi\varphi}=-\Gamma_1\Big(\varepsilon_{11}D^{111}_{\varphi o\varphi\dot{o}}H^{00}_{\varphi p\varphi\dot{p}}+\varepsilon_{33}\frac{1}{\gamma^2}D^{001}_{\varphi o\varphi\dot{o}}\cdot H^{11}_{\varphi p\varphi\dot{p}}\Big)-\Gamma_2\varepsilon_{33}n^2D^{00-1}_{\varphi o\varphi\dot{o}}H^{00}_{\varphi p\varphi\dot{p}}$$

$$\boldsymbol{M}_{uu}=\frac{\Gamma_1}{4}D^{001}_{uiu\bar{i}}H^{00}_{uju\dot{j}}$$

$$\boldsymbol{M}_{vv}=\frac{\Gamma_2}{4}D^{001}_{vkv\dot{k}}H^{00}_{vlv\dot{l}}$$

$$\boldsymbol{M}_{ww}=\frac{\Gamma_1}{4}D^{001}_{wmw\dot{m}}H^{00}_{wnw\dot{n}}$$

$$D_{\alpha\sigma\beta\dot{\sigma}}^{abc}=\int_{-1}^{1}\frac{\mathrm{d}^a P_\sigma(\bar{r})}{\mathrm{d}\bar{r}^a}\frac{\mathrm{d}^b P_{\bar{\sigma}}(\bar{r})}{\mathrm{d}\dot{r}^b}\left(\bar{r}+\delta\right)^c\mathrm{d}\bar{r}$$

$$H_{\alpha\tau\beta\bar{\tau}}^{ab}=\int_{-1}^{1}\frac{\mathrm{d}^a P_\tau(\bar{z})}{\mathrm{d}\bar{z}^a}\frac{\mathrm{d}^b P_{\dot{\tau}}(\bar{z})}{\mathrm{d}\bar{z}^b}\mathrm{d}\bar{z}$$

$$a,b=0,1;c=0,1,-1;\alpha,\beta=u,v,w$$

$$\sigma=i,k,m,o;\ \bar{\sigma}=\bar{i},\bar{k},\bar{m},\bar{o};$$

$$\tau=j,l,n,p;\ \bar{\tau}=\bar{j},\bar{l},\bar{n},\bar{p}. \tag{18}$$

where the elements of $\boldsymbol{K}$ and $\boldsymbol{M}$ are $[K_{\alpha\beta}]_{qr}$ and $[M_{\alpha\beta}]_{qr}$ with $q=\bar{I}(\sigma-1)+\tau$; $r=\bar{J}\left(\bar{\sigma}-1\right)+\bar{\tau}$;$\bar{I}=I$; K, M, $\bar{J}=J$, L, N.

By eliminating the electric potential from Eq.(16), the piezoelectrically stiffened eigenfrequency equation of the vibration is

$$\left(\begin{bmatrix}\tilde{\boldsymbol{K}}_{uu} & \tilde{\boldsymbol{K}}_{uv} & \tilde{\boldsymbol{K}}_{uw}\\ \tilde{\boldsymbol{K}}_{uv}^{\mathrm{T}} & \tilde{\boldsymbol{K}}_{uu} & \tilde{\boldsymbol{K}}_{uw}\\ \tilde{\boldsymbol{K}}_{uv}^{\mathrm{T}} & \tilde{\boldsymbol{K}}_{uv}^{\mathrm{T}} & \tilde{\boldsymbol{K}}_{ww}\end{bmatrix}-\bar{\Omega}^2\begin{bmatrix}\boldsymbol{M}_{uu} & 0 & 0\\ 0 & \boldsymbol{M}_{vv} & 0\\ 0 & 0 & \boldsymbol{M}_{ww}\end{bmatrix}\right)\begin{Bmatrix}\boldsymbol{A}\\ \boldsymbol{B}\\ \boldsymbol{C}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix} \tag{19}$$

where the piezoelectrically stiffened stiffness matrix is

$$\begin{bmatrix}\tilde{\boldsymbol{K}}_{uu} & \tilde{\boldsymbol{K}}_{uv} & \tilde{\boldsymbol{K}}_{uw}\\ \tilde{\boldsymbol{K}}_{uv}^{\mathrm{T}} & \tilde{\boldsymbol{K}}_{vv} & \tilde{\boldsymbol{K}}_{vw}\\ \tilde{\boldsymbol{K}}_{uv}^{\mathrm{T}} & \tilde{\boldsymbol{K}}_{uv}^{\mathrm{T}} & \tilde{\boldsymbol{K}}_{ww}\end{bmatrix}=\begin{bmatrix}\boldsymbol{K}_{uu} & \boldsymbol{K}_{uv} & \boldsymbol{K}_{uw}\\ \boldsymbol{K}_{uv}^{\mathrm{T}} & \boldsymbol{K}_{vv} & \boldsymbol{K}_{vw}\\ \boldsymbol{K}_{uv}^{\mathrm{T}} & \boldsymbol{K}_{uv}^{\mathrm{T}} & \boldsymbol{K}_{ww}\end{bmatrix}-\begin{bmatrix}\boldsymbol{K}_{u\varphi}\\ \boldsymbol{K}_{v\varphi}\\ \boldsymbol{K}_{w\varphi}\end{bmatrix}[\boldsymbol{K}_{\varphi\varphi}]^{-1}[\boldsymbol{K}_{u\varphi}^{\mathrm{T}}\quad \boldsymbol{K}_{v\varphi}^{\mathrm{T}}\quad \boldsymbol{K}_{w\varphi}^{\mathrm{T}}] \tag{20}$$

By solving Eq.(19) for eigenvalues, the analysis of free vibrations of the PZT ring can be performed to assist in the design of vital components. Specifically, the appropriate and optimal choices of key parameters will help in the selection of material and in other considerations such as ultrasonic power.

A complete analytical procedure for the vibrations of a PZT ring is presented with the Rayleigh-Ritz method. The electric field in the eigenvalue equation is eliminated for efficiency. The procedure and formulation are almost standard by adopting the sophisticated techniques and selection of displacement functions. The detailed procedure specifically developed for the PZT ring with targeted applications in ultrasonic welding will provide right tools in the design and improve the performance of transducers for the facial mark production lines in full capacity worldwide.

2 Simulation and Discussion

With the detailed process mentioned above, the procedure has to be tested and validated for the analysis. There are some pevious studies on similar structures with the Rayleigh-Ritz method and other numerical methods, and there are adequate references for the needed validation. The validation process is important in the analysis.

First, a completely free ring with inner-outer radius ratio $R_1/R_0=2$, $h/R_1=0.4$, and $\nu=0.3$ is analyzed. The material constants are $C_{11}=C_{33}=E(1-\nu)/[(1+\nu)(1-2\nu)]$, $C_{12}=C_{13}=\nu E/[(1+\nu)(1-2\nu)]$, $C_{44}=C_{66}=E/(2(1+\nu))$, where E is the Young's modulus. For comparison, the first non-dimensional frequencies of free vibrations, $\Omega=\omega R_1\sqrt{\rho/E}$, for wavenumber $n=1,2,3$ are calculated and compared in Table 1.

Table 1 The first non-dimensional frequency of circular annular plate

Method	$n=1$	$n=2$	$n=3$
This paper	1.943	0.691	1.679
Ref.[13]	1.943	0.691	1.680

Second, a ceramic PZT-4 ring with outer radius $R_1=25$ mm, inner radius $R_0=8.5$ mm, and thickness $h=7$ mm is analyzed for the frequency convergence in Table 2. For the calculation with the Rayleigh-Ritz method, orders of the Chebyshev polynomials are set as same as the circumferential wavenumber n. From the simulation results in Table 2, it is clear that with the 8th-order polynomials in each variable, the results are convergent.

Furthermore, the material properties of PZT-4 from the COMSOL material library are $C_{11}=$

139 GPa, $C_{12}=77.84$ GPa, $C_{13}=74.28$ GPa, $C_{33}=115.4$ GPa, $C_{44}=25.64$ GPa, $C_{66}=30.58$ GPa, $e_{15}=12.7\ \mathrm{C/m^2}$, $e_{31}=-5.2\ \mathrm{C/m^2}$, $e_{33}=15.1\ \mathrm{C/m^2}$, and $\varepsilon_{11}=762.5$, $\varepsilon_{33}=663.2$ with $\varepsilon_0=8.854\times10^{-12}$ F/m. The vibration analysis is performed with COMSOL for comparison in Table 3.

Table 2 Convergence check of resonant frequencies of a PZT-4 ring

Frequency / Hz	$I\times J$	$n=0$	$n=1$	$n=2$	$n=3$
f_1	8×8	14 764	25 307	7 886.2	18 952
	10×10	14 764	25 307	7 886.1	18 952
	20×20	14 764	25 307	7 886.1	18 952
f_2	8×8	37 194	38 630	18 644	39 595
	10×10	37 194	38 630	18 643	39 594
	20×20	37 194	38 630	18 643	39 594
f_3	8×8	76 988	80 291	45 469	67 417
	10×10	76 988	80 290	45 468	67 416
	20×20	76 988	80 289	45 468	67 416
f_4	8×8	123 777	89 272	54 521	73 994
	10×10	123 777	89 272	54 521	73 994
	20×20	123 777	89 272	54 521	73 994

Table 3 Comparison of vibration frequencies of a PZT-4 ring with finite element method (FEM)

Frequency / Hz	Method	$n=0$	$n=1$	$n=2$	$n=3$
f_1	This paper	14 764	25 307	7 886.2	18 952
	FEM	14 770	25 343	7 891.2	18 967
f_2	This paper	37 194	38 630	18 644	39 595
	FEM	37 200	38 630	18 652	39 622
f_3	This paper	76 988	80 291	45 469	67 417
	FEM	77 081	80 390	45 556	67 551
f_4	This paper	123 777	89 272	54 521	73 994
	FEM	123 830	89 277	54 522	74 007

In another example, vibration frequencies of a PZT ring are given in Table 4 with an outer radius of $R_1=25$ mm, inner radius of $R_0=8.5$ mm, height of $h=7$ mm. The material constants from COMSOL data library of PZT-8 are: $C_{11}=146.9$ GPa, $C_{12}=81.09$ GPa, $C_{13}=81.05$ GPa, $C_{33}=131.7$ GPa, $C_{44}=31.35$ GPa, $C_{66}=32.89$ GPa; $e_{15}=10.34\ \mathrm{C/m^2}$, $e_{31}=-3.875\ \mathrm{C/m^2}$, $e_{15}=13.91\ \mathrm{C/m^2}$; $\varepsilon_{11}=904.4$, $\varepsilon_{33}=561.6$, $\varepsilon_0=8.854\times10^{-12}$ F/m.

Table 4 Vibration frequencies of a PZT-8 ring in this paper and FEM

Frequency/ Hz	Method	$n=0$	$n=1$	$n=2$	$n=3$
f_1	This paper	14 921	25 678	8 095.3	19 393
	FEM	14 924	25 710	8 100.1	19 406
f_2	This paper	37 666	39 446	19 083	40 646
	FEM	37 666	39 446	19 092	40 674
f_3	This paper	76406	79 890	45 613	66 885
	FEM	76 461	79 974	45 684	66 987
f_4	This paper	124 544	91 402	55 866	75 920
	FEM	124 540	91 411	55 866	75 930

3 Conclusions

The Rayleigh-Ritz method is a highly efficient and reliable method for the calculation of resonant vibration frequencies of structures. Combined the equations of a PZT ring and the generalized displacement representation by Chebyshev polynomials, a simple and efficient procedure based on the Rayleigh-Ritz method for the calculation of vibration frequencies and mode shapes are presented and validated. The method has also been verified from analysis of FEM by using both PZT-4 and PZT-8 materials. The accurate and efficient calculation of vibration frequency is the first step in the selection of structural parameters for accurate frequency needed by the transducer. Further analysis and design parameters related to the ultrasonic power can also be calculated with the modification and extension of the Raleigh-

Ritz method presented in this paper. Moveover, the utilization of newer piezoelectric materials and different frequencies for welding and other processes can be accurately chosen with results from the method shown in this paper. We hope the proposed method, along with the codes in MATLAB can be freely available to engineers and students in the design for improvement of transducers needed in combating Covid-19 and related causes.

References

[1] WANG Ji, XIE Longtao. Essential concepts of piezoelectric ceramic disks for ultrasonic welding in facial mask production[EB/OL]. (2020-04-15) [2020-11-23]. http://piezo.nbu.edu.cn/info/1002/1849.htm.

[2] IKEGAMI S, UEDA I, KOBAYASHI S. Frequency spectra of resonant vibration in disk plates of PbTi03 piezoelectric ceramics[J]. The Journal of the Acoustical Society of America, 1973 (55): 339-344.

[3] CUI Xiuting, LIANG Jiahui, WU Mingxin. The study of the radial-axial coupling vibration for piezoelectric-ceramics-cylindrical-vibrator[J]. Acta Aeronautic et Astronautica Sinica, 1989, 10(10): B545-B552.

[4] ZHOU Ding, LO S H, AU F T K, et al. 3-D vibration analysis of skew thick plates using Chebyshev-Ritz method[J]. International Journal of Mechanical Sciences, 2006 (48): 1481-1493.

[5] GIEBE E, BLECHSCHMIDT E. Experimental and theoretical studies of extensional vibrations of rods and tubes[J]. Annals of Physics, 1933 (18): 417-485.

[6] LIU Shuyu. Analysis of the equivalent circuit of piezoelectric ceramic disk resonators in coupled vibration[J]. Journal of Sound and Vibration, 2000, 231 (2): 277-290.

[7] DING Haojiang, JIANG Aimin. A boundary integral formulation and solution for 2D problems in magneto-electro-elastic media[J]. Computers and Structures, 2004 (82): 1599-1607.

[8] LIU Shuyu. The equivalent circuit of coupled vibration for piezoelectric ceramic disks[J]. Piezoelectrics and Acoustooptics, 1993, 15(6): 32-37.

[9] WANG Xu, ZHONG Zheng. A finitely long circular cylindrical shell of piezoelectric/piezomagnetic composite under pressuring and temperature change[J]. International Journal of Engineering Science, 2003 (41): 2429-2445.

[10] QIANG Panfu. Coupling vibration of finite-size piezoelectric ceramic plates[J]. Acta Acustica, 1984, 9 (1): 47-54.

[11] ZHOU Ding, YAU KAI C, SH L, et al. 3D vibration analysis of solid and hollow circular cylinders via Chebyshev-Ritz method[J]. Comput Methods Appl Mech Engrg, 2003 (192): 1575-1589.

[12] DONG Chunying. Vibration of electro-elastic versus magneto-elastic circular / annular plates using the Chebyshev-Ritz method[J]. Journal of Sound and Vibration, 2008 (317): 219-235.

[13] SO J, LEISSA A W. Three-dimensional vibrations of thick circular and annular plates[J]. Journal of Sound and Vibration, 1998 (209): 15-41.

Acknowledgements This work was supported in part by the National Natural Science Foundation of China (No. 11672142) and the Technology Innovation 2025 Program of the Municipality of Ningbo (No. 2019B10122). The corresponding author is indebted to Prof. LIN Shuyu, Institute of Applied Acoustics, Shaanxi Normal University, for valuable information on transducer materials and configuration.

Authors Ms. **WU Jinghui** received her M.S. degree in Mechanical Engineering from Ningbo University in 2015. From 2015, she has been a Ph.D. student in the School of Mechanical Engineering and Mechanics, Ningbo University, China.

Prof. **WANG Ji** received the B.S. and Ph.D. degrees from Princeton University, the USA, in 1993 and 1996, respectively. From 2002 to present, he has been a Qianjiang Chair Professor, Ningbo University, China. He is also the funding director of the Piezoelectric Devices Laboratory. His research has focused on the analysis of acoustic wave resonators.

Author contributions Prof. **WANG Ji** initiated the study and completed the manuscript. Mr. **ZHAO Shengquan** contributed analysis of the vibrations with helps from Ms. **WU Jinghui** and Dr. **XIE Longtao**. All authors checked and commented on the manuscript draft and approved the submission.

Competing interests The authors declare no competing interests.

(Production Editor: ZHANG Tong)

压电陶瓷管在口罩焊接中作为超声换能器的振动频率分析

吴景慧，赵胜权，谢龙涛，王 骥

（宁波大学机械工程与力学学院压电技术实验室，宁波 315211，中国）

摘要：Covid-19爆发后，口罩作为重要个人防护装备，其需求呈爆炸式增长，挑战相关制造业的技术和产能。生产口罩焊接过程中对压电陶瓷换能器的需求亦大幅增加。为保证陶瓷换能器能满足快速生产所需要的技术要求，本文采用瑞利-里茨方法对压电陶瓷进行振动分析，得到最简单的解，并通过选择不同的材料，分析对比它们达到所需频率和能量标准的重要参数。所提出的方法和最终获得的数据结果可供工程师快速开发压电陶瓷传感器，从而在口罩焊接或其他类似材料的焊接中达到标准要求。

关键词：压电陶瓷；传感器；口罩焊接；个人防护装备