Computationally Efficient 2D-DOA Estimation for Uniform Planar Arrays: RD-ROOT-MUSIC Algorithm

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Abstract: The problem of two-dimensional direction of arrival (2D-DOA) estimation for uniform planar arrays (UPAs) is investigated by employing the reduced-dimensional (RD) polynomial root finding technique and 2D multiple signal classification (2D-MUSIC) algorithm. Specifically, based on the relationship between the noise subspace and steering vectors, we first construct 2D root polynomial for 2D-DOA estimates and then prove that the 2D polynomial function has infinitely many solutions. In particular, we propose a computationally efficient algorithm, termed RD-ROOT-MUSIC algorithm, to obtain the true solutions corresponding to targets by RD technique, where the 2D root-finding problem is substituted by two one-dimensional (1D) root-finding operations. Finally, accurate 2D-DOA estimates can be obtained by a sample pairing approach. In addition, numerical simulation results are given to corroborate the advantages of the proposed algorithm.

Key words: uniform planar array (UPA); direction of arrival (DOA) estimation; RD-ROOT-MUSIC algorithm

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0 Introduction

In the past few decades, two-dimensional direction of arrival (2D-DOA) estimation has drawn considerable attention and has been utilized in many fields such as astronomy, wireless communications, and radar systems[1-3]. Most of the existing research mainly focus on structures of 2D arrays such as uniform planar array (UPA)[4-6], L-shaped array[7-8], uniform circular array (UCA)[9] and two parallel uniform linear arrays[10]. For these 2D arrays, the spacing of adjacent sensors should be no larger than half wavelength to eliminate phase ambiguity problem.

Conventional 2D-DOA methods mainly include two categories that one is based on spectrum peak search[11-18], and the other exploits the rotation invariance algorithm[9,19-21]. In Ref.[11], the traditional 2D multiple signal classification (2D-MUSIC) algorithm is employed to UPA, where the well-performed DOA estimates can be obtained with expensive computational complexity caused by the total spectral search (TSS). Ref.[17] combined the reduced-dimensional MUSIC (RD-MUSIC) method with multiple-input-multiple-out (MIMO) radar to complete 2D-DOA estimation, to further reduce the computational complexity. The rotation invariance algorithm can directly calculate the DOA estimates and avoid spectral peak searching, but cannot achieve as good DOA estimates as the algorithms based on spectral peak searching. Ref.[18] proposed one-dimensional (1D) partial spectral search approach for coprime planar arrays, which achieved enhanced DOA performance with much

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lower computational cost as compared with 2D-MUSIC algorithm, but it still performed peaks searching on spectrum function. To achieve a better tradeoff between computational complexity and DOA estimation performance, scholars have proposed some methods based on MUSIC algorithm to reduce computational complexity by utilizing polynomial root finding technique[22-24]. Ref.[23] proposed a polynomial rooting algorithm based on L-shaped array, which failed to make full use of 2D array information. In Refs.[24-25], the initial 1D roots were calculated first and then were used to obtain all the DOA estimates based on the spectrum function. However, the 2D-DOA estimation performance of the methods degrades at low signal-to-noise ratio (SNR) because bad initial DOA estimates exist in this case.

Considering the disadvantages mentioned above in conventional methods, we propose a computationally efficient algorithm based on RD polynomial root finding technique for UPA termed RD-ROOT-MUSIC algorithm. Based on the 2D-MUSIC spectrum function, we consider 2D-ROOT-MUSIC polynomial firstly and explore to find two paired solutions corresponding to targets directly. Subsequently, we prove that 2D polynomial contains infinitely many solutions, and it is impossible for us to obtain the solution to a bivariate higher-order equation by once polynomial root finding without other qualifications. According to 2D-ROOT-MUSIC polynomial, we construct two 1D root polynomials to obtain 2D-DOA estimates corresponding to targets from infinitely many solutions by RD root finding technique. Furthermore, we also exploit the feasibility and effectiveness of the RD process. Although the proposed algorithm requires an additional parameter pairing procedure, it only needs extremely low complexity. Compared with RD-MUSIC algorithm and 2D-MUSIC algorithm, the proposed algorithm can get roughly the same 2D-DOA estimation accuracy, but the computational complexity is much lower. Numerical simulations corroborate the effectiveness and priority of the proposed algorithm.

Specially, the main contributions of this paper are summarized as follows:

(1) We devise the 2D-DOA estimation problem for UPA as a polynomial rooting problem, to achieve high accuracy DOA estimates with a much lower computational cost.

(2) We prove that 2D-ROOT-MUSIC polynomial contains infinitely many solutions.

(3) We propose a computationally efficient algorithm termed RD-ROOT-MUSIC algorithm, which exploits the feasibility and effectiveness of the RD polynomial root finding process.

Notations: Bold lower-case and upper-case characters are used to represent the vectors and matrices, respectively. $(\cdot)^{\mathrm{T}}$, $(\cdot)^{\mathrm{H}}$, $(\cdot)^{-1}$ and $(\cdot)^{*}$ represent transpose, conjugate transpose, inverse and conjugate operations, respectively. $\odot$ and $\otimes$ stand for the Khatri-Rao product and Kronecker product, respectively. $\boldsymbol{I}_M$ is the $M\times M$ identity matrix. angle$(\cdot)$ stands for the phase operator and det$(\cdot)$ denotes the determinant of the matrix.

1 Data Model

The UPA structure is shown in Fig.1, which contains $M\times N$ sensors. The inter-element spacing is $d=\lambda/2$, where λ denotes the wavelength.

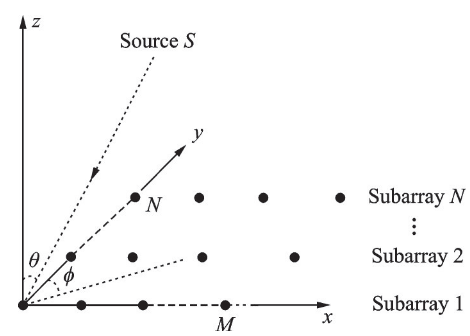


Fig.1 UPA structure

Assume there are $K(K<MN)$ narrowband far-field uncorrelated sources impinging on the UPA from 2D-DOA elevation angles (θ_k,ϕ_k), $k=1,2,\cdots,K$, where θ_k and ϕ_k are the elevation and azimuth angles of the kth sources $(\theta_k\in[0,\pi/2]$,

$\phi_k \in [0, \pi]$), respectively.

The received signals can be expressed as[11]

$$X = AS + N \tag{1}$$

where $X = [x(1), x(2), \cdots, x(L)]$ and L represents the number of snapshots, $S = [s_1, s_2, \cdots, s_K]^{\mathrm{T}}$ is the source signal matrix and $s_k = [s_k(1), s_k(2), \cdots, s_k(L)]$, and $s_k(l) = \beta_k e^{j2\pi f_k} t$ with the Doppler frequency f_k and the amplitude β_k. $N \in \mathbf{C}^{MN \times L}$ is the additive white Gaussian noise matrix with zero mean and variance σ^2. $A \in \mathbf{C}^{MN \times K}$ is the steering matrix of the UPA and

$$A = A_y \odot A_x = [a_y(\theta_1, \phi_1) \otimes a_x(\theta_1, \phi_1), \cdots, a_y(\theta_K, \phi_K) \otimes a_x(\theta_K, \phi_K)] \tag{2}$$

where $a_x(\theta_k, \phi_k)$ and $a_y(\theta_k, \phi_k)$ are steering vectors of A. They can be represented by

$$a_x(\theta_k, \phi_k) = [1, e^{j2\pi d \cos\phi_k \sin\theta_k/\lambda}, \cdots, e^{j2\pi(M-1)d\cos\phi_k \sin\theta_k/\lambda}]^{\mathrm{T}} \tag{3}$$

$$a_y(\theta_k, \phi_k) = [1, e^{j2\pi d \sin\phi_k \sin\theta_k/\lambda}, \cdots, e^{j2\pi(M-1)d\sin\phi_k \sin\theta_k/\lambda}]^{\mathrm{T}} \tag{4}$$

The covariance matrix can be calculated with L snapshots by

$$\hat{R} = \frac{1}{L}\sum_{l=1}^{L} x(l) x^{\mathrm{H}}(l) \tag{5}$$

Based on the eigenvalue decomposition (EVD), $\hat{R}$ can be decomposed as

$$\hat{R} = E_s D_s E_s^{\mathrm{H}} + E_n D_n E_n^{\mathrm{H}} \tag{6}$$

where E_s is formed by the eigenvectors corresponding to the maximum K eigenvalues, and E_n is composed of the rest eigenvectors. D_s and D_n are diagonal matrices. The diagonal elements of D_s are made up of the largest K eigenvalues, and the diagonal elements of D_n are composed of other eigenvalues.

Remark 1 In this paper, it is assumed that K is the prior information which can be estimated by the methods in Refs.[26-27].

2 The Proposed Algorithm

In this section, we first construct 2D-ROOT-MUSIC polynomial and prove that it contains infinitely many solutions, then we apply the RD polynomial root finding process to get two 1D polynomials. Besides, we exploit the feasibility and effectiveness of the RD process. Accurate 2D-DOA estimates corresponding to targets can be obtained by the proposed algorithm.

2.1 2D-MUSIC algorithm

The spectrum function of UPA can be expressed as[11]

$$P_{\text{2D-MUSIC}}(\theta, \phi) = \frac{1}{[a_y(\theta, \phi) \otimes a_x(\theta, \phi)]^{\mathrm{H}} E_n E_n^{\mathrm{H}} [a_y(\theta, \phi) \otimes a_x(\theta, \phi)]} \tag{7}$$

where $a_y(\theta, \phi) = [1, e^{j2\pi d \sin\phi \sin\theta/\lambda}, \cdots, e^{j2\pi(M-1)d\sin\phi\sin\theta/\lambda}]^{\mathrm{T}}$, and $a_x(\theta, \phi) = [1, e^{j2\pi d \cos\phi \sin\theta/\lambda}, \cdots, e^{j2\pi(N-1)d\cos\phi\sin\theta/\lambda}]^{\mathrm{T}}$.

In Ref.[12], the authors performed 2D partial spectral search (PSS) on Eq.(7) to obtain 2D-DOA estimates, which requires a time-consuming 2D spectral search. In Ref.[14], Zhang et al. utilized RD technique, but it still needed 1D spectral search. We utilize polynomial root finding process instead of spectral peak search, which greatly reduces the computational cost.

2.2 2D-ROOT-MUSIC polynomial

Define

$$\begin{cases} u = \sin\theta \sin\phi \\ v = \cos\theta \sin\phi \end{cases} \tag{8}$$

then, the steering vectors can be rewritten as

$$a_y(\theta, \phi) = a_y(u) = [1, e^{j2\pi du/\lambda}, \cdots, e^{j2\pi d(N-1)u/\lambda}]^{\mathrm{T}} \tag{9}$$

$$a_x(\theta, \phi) = a_x(v) = [1, e^{j2\pi dv/\lambda}, \cdots, e^{j2\pi d(M-1)v/\lambda}]^{\mathrm{T}} \tag{10}$$

2D-ROOT-MUSIC polynomial can be expressed as

$$V(u, v) = [a_y(u) \otimes a_x(v)]^{\mathrm{H}} E_n E_n^{\mathrm{H}} [a_y(u) \otimes a_x(v)] = 0 \tag{11}$$

Lemma 1 The UPA structure is shown in Fig.1. By performing EVD on covariance matrix, 2D-ROOT-MUSIC polynomial corresponding to the UPA can be expressed as Eq.(11), which contains infinitely many solutions.

Proof See Appendix A.

According to Lemma 1, it is not feasible to obtain the paired u and v directly from 2D-ROOT-MUSIC polynomial because of infinitely many solutions. We use RD polynomial root finding technique to solve this problem.

2.3 RD polynomial root finding process

According to Eq.(11), $V(u,v)$ can be reconstructed as[14,17]

$$V(u,v)=\boldsymbol{a}_x^{\mathrm{H}}(v)\left[\boldsymbol{a}_y(u)\otimes\boldsymbol{I}_M\right]^{\mathrm{H}}\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}\left[\boldsymbol{a}_y(u)\otimes\boldsymbol{I}_M\right]\boldsymbol{a}_x(v)=\boldsymbol{a}_x^{\mathrm{H}}(v)\boldsymbol{Q}(u)\boldsymbol{a}_x(v)\tag{12}$$

or

$$V(u,v)=\boldsymbol{a}_y^{\mathrm{H}}(u)\left[\boldsymbol{I}_N\otimes\boldsymbol{a}_x(v)\right]^{\mathrm{H}}\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}\left[\boldsymbol{I}_N\otimes\boldsymbol{a}_x(v)\right]\boldsymbol{a}_y(u)=\boldsymbol{a}_y^{\mathrm{H}}(u)\boldsymbol{Q}(v)\boldsymbol{a}_y(u)\tag{13}$$

where

$$\boldsymbol{Q}(u)=\left[\boldsymbol{a}_y(u)\otimes\boldsymbol{I}_M\right]^{\mathrm{H}}\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}\left[\boldsymbol{a}_x(v)\otimes\boldsymbol{I}_M\right]\tag{14}$$

$$\boldsymbol{Q}(v)=\left[\boldsymbol{I}_N\otimes\boldsymbol{a}_x(v)\right]^{\mathrm{H}}\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}\left[\boldsymbol{I}_N\otimes\boldsymbol{a}_x(v)\right]\tag{15}$$

Basically, the above states derivations covert DOA estimation into polynomial root finding with Eq.(11), if $\mathrm{e}^{\mathrm{j}2\pi du/\lambda}$ does not correspond to one target and if[24-25]

$$\mathrm{Rank}(\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}})=MN-K\geqslant N\Rightarrow N\leqslant M(N-1)\tag{16}$$

Considering about the noise with non-zero value, the matrix $\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}$ is invertible, and the determinant is not equal to zero.

Eq.(16) means that $\det\{\boldsymbol{Q}(u)\}$ is non-zero polynomial. Obviously, $\boldsymbol{Q}(u)$ is a factor of $V(u,v)$. Since $\boldsymbol{Q}(u)$ contains only the variable u, if u takes the value that satisfies

$$\det\{\boldsymbol{Q}(u)\}=0\tag{17}$$

the roots of Eq.(17) must contain u corresponding to targets. Eq.(17) also means

$$V(u,v)=\boldsymbol{a}_x^{\mathrm{H}}(v)\boldsymbol{Q}(u)\boldsymbol{a}_x(v)=0\tag{18}$$

The analysis of the variable v is similar. Then the problem of obtaining u and v corresponding to targets from infinitely many solutions in Eq.(11) can be converted into two 1D polynomials root finding problems. Eqs.(12) and (13) can be converted into

$$\det\{\boldsymbol{Q}(u)\}=\det\{\left[\boldsymbol{a}_y(u)\otimes\boldsymbol{I}_M\right]^{\mathrm{H}}\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}\left[\boldsymbol{a}_y(u)\otimes\boldsymbol{I}_M\right]\}=0\tag{19}$$

$$\det\{\boldsymbol{Q}(v)\}=\det\{\left[\boldsymbol{I}_N\otimes\boldsymbol{a}_x(v)\right]^{\mathrm{H}}\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}\left[\boldsymbol{I}_N\otimes\boldsymbol{a}_x(v)\right]\}=0\tag{20}$$

Define

$$\begin{cases}z_1=\mathrm{e}^{\mathrm{j}2\pi du/\lambda}\\ z_2=\mathrm{e}^{\mathrm{j}2\pi dv/\lambda}\end{cases}\tag{21}$$

The steering vectors can be rewritten as

$$\boldsymbol{a}_y(u)=\left[1,\mathrm{e}^{\mathrm{j}2\pi du/\lambda},\cdots,\mathrm{e}^{\mathrm{j}2\pi(N-1)du/\lambda}\right]^{\mathrm{T}}=\left[1,z_1,\cdots,z_1^{N-1}\right]^{\mathrm{T}}=\boldsymbol{a}_y(z_1)\tag{22}$$

$$\boldsymbol{a}_x(v)=\left[1,\mathrm{e}^{\mathrm{j}2\pi dv/\lambda},\cdots,\mathrm{e}^{\mathrm{j}2\pi(M-1)dv/\lambda}\right]^{\mathrm{T}}=\left[1,z_2,\cdots,z_2^{M-1}\right]^{\mathrm{T}}=\boldsymbol{a}_x(z_2)\tag{23}$$

To eliminate u^* and v^*, we can replace $\left[\boldsymbol{a}_y(u)\otimes\boldsymbol{I}_M\right]^{\mathrm{H}}$ with $z_1^{N-1}\left[\boldsymbol{a}_y^{\mathrm{T}}(z_1^{-1})\otimes\boldsymbol{I}_M\right]^{\mathrm{H}}$ and replace $\left[\boldsymbol{I}_N\otimes\boldsymbol{a}_x(v)\right]^{\mathrm{H}}$ with $z_2^{M-1}\left[\boldsymbol{I}_N\otimes\boldsymbol{a}_x^{\mathrm{T}}(z_2^{-1})\right]^{\mathrm{H}}$, i.e.

$$\det\{\boldsymbol{Q}(z_1)\}=\det\{z_1^{N-1}\left[\boldsymbol{a}_y^{\mathrm{T}}(z_1^{-1})\otimes\boldsymbol{I}_M\right]^{\mathrm{H}}\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}\left[\boldsymbol{a}_y(z_1)\otimes\boldsymbol{I}_M\right]\}=0\tag{24}$$

$$\det\{\boldsymbol{Q}(z_2)\}=\det\{z_2^{M-1}\left[\boldsymbol{I}_N\otimes\boldsymbol{a}_x^{\mathrm{T}}(z_2^{-1})\right]^{\mathrm{H}}\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}\left[\boldsymbol{I}_N\otimes\boldsymbol{a}_x(z_2)\right]\}=0\tag{25}$$

Since the degree of $\det\{\boldsymbol{Q}(z_1)\}$ and $\det\{\boldsymbol{Q}(z_2)\}$ are even, we can take roots $\hat{z}_{11},\cdots,\hat{z}_{1k},\cdots,\hat{z}_{1K}$ of Eq.(24) with the largest K amplitudes in the unit circle to obtain estimates of $\hat{u}_k$, and take roots $\hat{z}_{21},\cdots,\hat{z}_{2i},\cdots,\hat{z}_{2K}$ of Eq.(25) with the largest K amplitudes in the unit circle to obtain estimates of $\hat{v}_i$.

$$\hat{u}_k=(\mathrm{angle}(\hat{z}_{1k})\lambda/2\pi d)\quad k=1,\cdots,K\tag{26}$$

$$\hat{v}_i=(\mathrm{angle}(\hat{z}_{2i})\lambda/2\pi d)\quad i=1,\cdots,K\tag{27}$$

2.4 Pairing and 2D-DOA estimation

The estimates of $\hat{u}_k$ and $\hat{v}_i$ are separated, so it is necessary to pair $\hat{u}_k$ and $\hat{v}_i$ to complete 2D-DOA estimation. Construct cost function ($k=1,\cdots,K$)

$$V_{k,i}=\underset{i=1,\cdots,K}{\arg\min}\left\|\left[\boldsymbol{a}_y(\hat{u}_k)\otimes\boldsymbol{a}_x(\hat{v}_i)\right]^{\mathrm{H}}\cdot\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}\left[\boldsymbol{a}_y(\hat{u}_k)\otimes\boldsymbol{a}_x(\hat{v}_i)\right]\right\|\tag{28}$$

where $\boldsymbol{a}_y(\hat{u}_k)$ and $\boldsymbol{a}_x(\hat{v}_i)$ represent steering vectors reconstructed by $\hat{u}_k$ and $\hat{v}_i$, respectively.

It is obvious that the total number of $V_{k,i}$ is K^2. Then for each k, we can calculate $V_{k,i}(1\leqslant i\leqslant K)$ to select the minimum value and return corresponding i into i'. 2D-DOA estimates can be calculated by

$$\hat{\theta}_k=\arcsin\left(\sqrt{\hat{u}_k^2+\hat{v}_{i'}^2}\right)\quad 1\leqslant k\leqslant K\tag{29}$$

$$\hat{\phi}_k=\arctan\left(\hat{u}_k/\hat{v}_{i'}\right)\quad 1\leqslant k\leqslant K\tag{30}$$

where $\hat{v}_{i'}$ is reconstructed by $i'(1\leqslant i'\leqslant K)$.

2.5 Detailed steps of the proposed algorithm

The major steps of the proposed algorithm to obtain 2D-DOA estimates for the UPA are given as follows.

Step 1 Compute the covariance matrix $\hat{\boldsymbol{R}}$ and perform EVD to obtain noise subspace $\boldsymbol{E}_n$.

Step 2 Reconstruct $V(u, v)$ according to Eq.(11).

Step 3 Perform RD process on $V(u, v)$ to get Eqs.(24) and (25).

Step 4 Calculate $\hat{u}_k$ and $\hat{v}_i$ according to Eqs.(26) and (27).

Step 5 Complete pairing procedure and obtain the 2D DOA estimation of targets according to Eqs.(29) and (30).

3 Performance Analysis

3.1 Complexity analysis

We analyze the computational complexity of the proposed algorithm and compare it with the 2D-ESPRIT algorithm[9], 2D-MUSIC algorithm[11], and RD-MUSIC algorithm[17]. For the proposed algorithm, calculating the covariance matrix needs $O\{(MN)^2L\}$, eigenvalue decomposition requires $O\{(MN)^3\}$, polynomial root finding costs $O\{(2N(M-1))^3+(2M(N-1))^3\}$ and the pairing process requires $O\{K^2(MN+1)(MN-K)\}$, so the total computational complexity of the proposed algorithm is

$$O\{(MN)^2L+(MN)^3+(2N(M-1))^3+ (2M(N-1))^3+K^2(MN+1)(MN-K)\}$$

The computational complexities of 2D-ESPRIT with UCA, 2D-MUSIC with passive array and RD-MUSIC with MIMO radar are given in Refs.[9], [11] and [17], respectively. We extend them to the complexities of UPA for comparison. Both algorithms require $O\{(MN)^2L+(MN)^3\}$ to obtain the covariance matrix and perform eigenvalue decomposition. The peak search of 2D-MUSIC and RD-MUSIC costs $O\{n_1^2(MN+1)(MN-K)\}$ and $O\{n_2K(M^2N+M^2)(MN-K)\}$, respectively, where $n_1=90°/\Delta$ and $n_2=2°/\Delta$ represent the search times, and the peak search interval is $\Delta=0.01°$. For 2D-ESPRIT algorithm, further accurate estimation needs $O\{2K^2(M-1)N+6K^3+2K^2(N-1)M\}$. The computational complexity of different algorithms mentioned above is summarized in Table 1. Besides, Fig.2 shows the computational complexity comparison of different algorithms versus different N, where $M=6, K=2, L=500$. The complexity comparison versus snapshot is depicted in Fig.3, where $M=6, N=6, K=2$. It is observed from Figs.2 and 3 that the proposed algorithm outperforms the 2D-MUSIC and RD-MUSIC algorithms in computational complexity, and its computational complexity is approximately as low as the 2D-ESPRIT algorithm, proving the computational efficiency of the proposed algorithm.

Table 1 Complexity comparison of different methods

Algorithm	Computational complexity
Proposed	$O\{(MN)^2L+(MN)^3+(2N(M-1))^3+(2M(N-1))^3+K^2(MN+1)(MN-K)\}$
2D-ESPRIT[9]	$O\{L(MN)^2+(MN)^3+2K^2(M-1)N+6K^3+2K^2(N-1)M\}$
2D-MUSIC[11]	$O\{L(MN)^2+(MN)^3+n_1^2[(MN+1)(MN-K)]\}$
RD-MUSIC[17]	$O\{(MN)^2L+(MN)^3+n_2K(M^2N+M^2)(MN-K)\}$

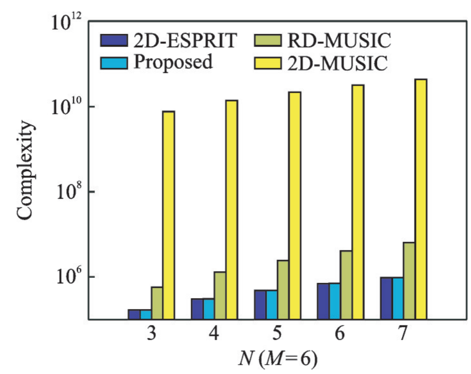


Fig.2 Computational complexities of different methods versus different N

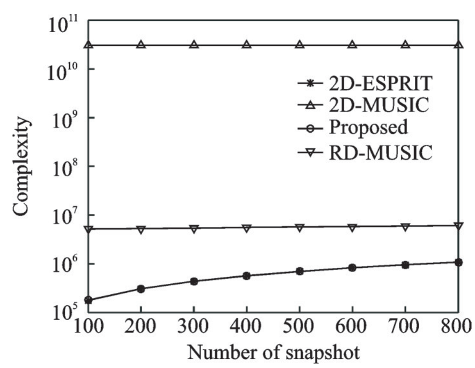


Fig.3 Computational complexities of different methods versus different snapshots

3.2 Cramer-Rao bound

The Cramer-Rao bound (CRB) represents a lower bound to the error variance of parameter estimator. We give the derivation of CRB for UPA to evaluate the 2D-DOA estimation performance as follows

Define $\boldsymbol{A}=[\boldsymbol{a}_y(\theta_1,\phi_1)\otimes\boldsymbol{a}_x(\theta_1,\phi_1),\cdots,\boldsymbol{a}_y(\theta_K,\phi_K)\otimes\boldsymbol{a}_x(\theta_K,\phi_K)]$.

According to Ref.[28], the CRB of UPA can be given by

$$\mathrm{CRB}=\frac{\sigma^2}{2L}\{\mathrm{Re}[\boldsymbol{D}^{\mathrm{H}}\boldsymbol{\Pi}_A^{\perp}\boldsymbol{D}\oplus\hat{\boldsymbol{P}}^{\mathrm{T}}]\}^{-1} \tag{31}$$

where $\boldsymbol{D}=\left[\frac{\partial\boldsymbol{a}_1}{\partial\theta_1},\cdots,\frac{\partial\boldsymbol{a}_k}{\partial\theta_k},\cdots,\frac{\partial\boldsymbol{a}_K}{\partial\theta_K},\frac{\partial\boldsymbol{a}_1}{\partial\phi_1},\cdots,\frac{\partial\boldsymbol{a}_k}{\partial\phi_k},\cdots,\frac{\partial\boldsymbol{a}_K}{\partial\phi_K}\right]$, $\hat{\boldsymbol{P}}=\begin{bmatrix}\hat{\boldsymbol{P}}_s & \hat{\boldsymbol{P}}_s\\ \hat{\boldsymbol{P}}_s & \hat{\boldsymbol{P}}_s\end{bmatrix}$, $\hat{\boldsymbol{P}}_s=\boldsymbol{S}\boldsymbol{S}^{\mathrm{H}}/L$, $\boldsymbol{\Pi}_A^{\perp}=\boldsymbol{I}_{M\times N}-\boldsymbol{A}[\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}]^{-1}\cdot\boldsymbol{A}^{\mathrm{H}}$. Here L represents the number of snapshots, $\boldsymbol{a}_k$ the kth column of $\boldsymbol{A}$, and σ^2 the variance of the received noise.

3.3 Advantages

Based on the above discussions, the proposed algorithm has the following advantages.

(1) The proposed algorithm converts 2D polynomial into 1D polynomials, and does not require spectrum search, thus reducing the computational complexity significantly.

(2) The complexity of the proposed algorithm is approximately as low as the 2D-ESPRIT algorithm, which is much lower than the 2D-MUSIC and RD-MUSIC algorithms.

(3) The 2D-DOA estimation accuracy of proposed algorithm is significantly better than 2D-PM and 2D-ESPRIT algorithms, and is almost the same as 2D-MUSIC and RD-MUSIC algorithms.

(4) The proposed algorithm can be effectively used for 2D-DOA estimation with high accuracy 2D-DOA estimation and is also attractive in massive MIMO radar system.

4 Simulation Results

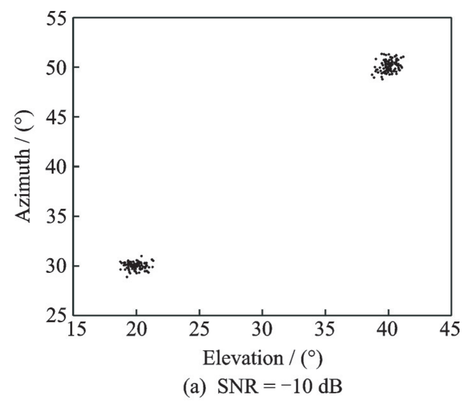
In this section, we employ Monte Carlo simulation to simulate the proposed algorithm. The number of Monte Carlo simulation is 500. Assume that the incident angles are $(\theta_1,\phi_1)=(20°,30°)$ $(\theta_2,\phi_2)=(40°,50°)$ of two unrelated sources (i.e. $K=2$). Inter-element spacing d equals half wavelength. We can define the root mean square error (RMSE) of the 2D-DOA estimation as

$$\mathrm{RMSE}=\frac{1}{K}\sum_{k=1}^{K}\sqrt{\frac{1}{500}\sum_{i=1}^{500}(\hat{\phi}_{k,i}-\phi_k)^2+(\hat{\theta}_{k,i}-\theta_k)^2} \tag{32}$$

where ϕ_k and θ_k represent the true azimuth and elevation of the kth target, respectively. $\hat{\phi}_{k,i}$ and $\hat{\theta}_{k,i}$ are estimated value of ϕ_k and θ_k in the ith Monte Carlo simulation, respectively.

4.1 Scatter figures

Fig.4 shows the scatter figures of the proposed algorithm, where $M=6$ and $N=6$. The SNR of Fig.4(a) and Fig.4(b) are -10 dB and 10 dB, respectively. It is obviously shown that the proposed algorithm can obtain paired azimuth and elevation effectively.



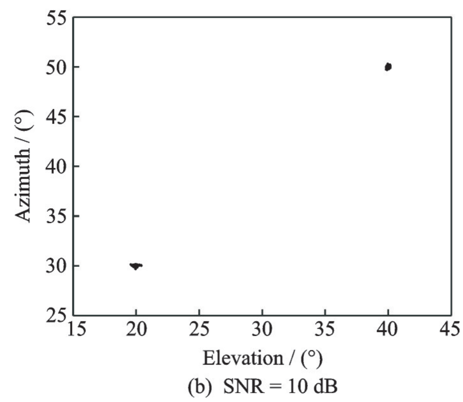


Fig.4 Scatter figures of the proposed algorithm

4.2 RMSE performance comparison results versus snapshot

Fig.5 gives the comparison results of RMSE performance for UPA versus snapshot, where $M=6, N=6$. An increased number of snapshots means more sampling data. As the number of snapshots increases, it is illustrated clearly that the 2D-DOA estimation performance gets better in Fig.5.

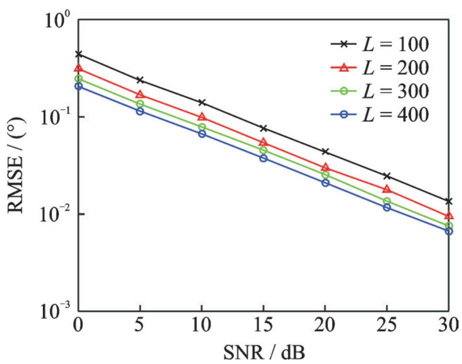


Fig.5 RMSE performance versus snapshot

4.3 RMSE performance comparison results versus sensor

Fig.6 depicts the comparison results of RMSE performance versus sensors M and N, where $L=200$. An increased number of sensors means more diversity gain received by UPA. Fig.6 indicates that the 2D-DOA estimation performance is enhanced as the number of sensors increases.

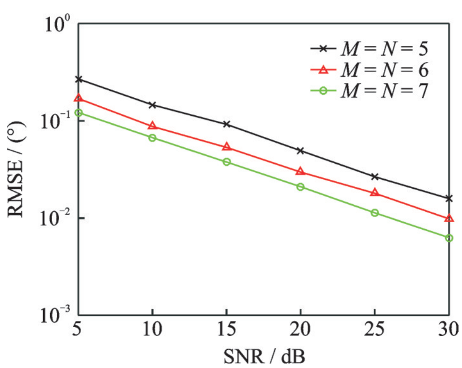


Fig.6 RMSE performance versus sensors

4.4 2D-DOA performance comparison results of different algorithms

We investigate 2D-DOA estimation performance comparison results of the proposed algorithm, 2D-MUSIC algorithm, 2D-PM algorithm[21], 2D RD-MUSIC and 2D-ESPRIT algorithms for UPA, where $M=6, N=6$, SNR$=10$ dB in Fig.7(a) and $M=6, N=6, L=200$ in Fig.7(b). The range of spectrum searching for 2D-MUSIC is $(0°, 90°)$, and searching interval is $\Delta=0.01°$. As depicted in Fig.7, the performance of the proposed algorithm is almost identical to the 2D-MUSIC algorithm and RD-MUSIC algorithm. Compared with the 2D-ESPRIT and 2D-PM algorithms, the proposed algorithm performs significantly better, verifying its effectiveness.

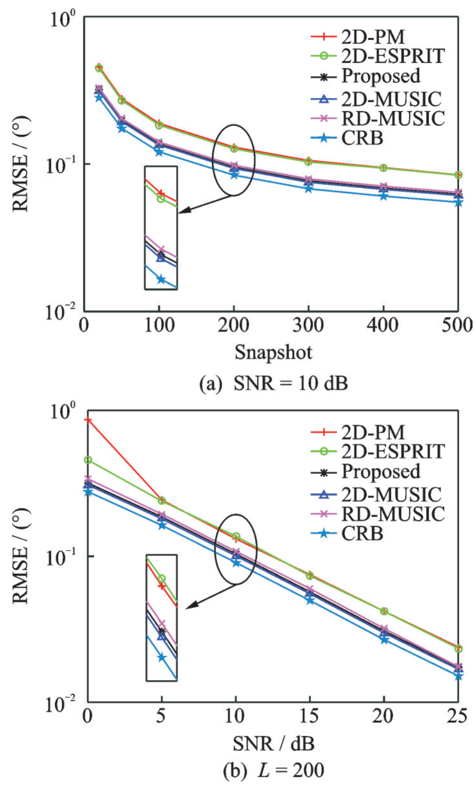


Fig.7 Comparison of RMSE performance of different algorithms

5 Conclusions

Based on RD polynomial rooting technique and 2D-MUSIC algorithm, a novel computationally efficient algorithm with improved DOA estimation performance is proposed in this paper. We first take 2D ROOT-MUSIC polynomial into consideration, which contains infinitely many solutions, and it is difficult to get paired solutions directly. For further, a novel method termed RD-ROOT-MUSIC algorithm with UPA is proposed to solve this problem. It converts 2D polynomial into two 1D polynomials, then the computational complexity and root finding difficulty can be reduced, effectively. In addition, a better trade-off between computational complexity and 2D-DOA performance is achieved, as compared with other existing methods. Numerical simulations validate the effectiveness and efficiency of the proposed algorithm.

Appendix A

Fundamental theorem of algebra: Every non-constant single-variable polynomial with complex coefficients has at least one complex root.

According to Eqs.(11—13), 2D-ROOT-MUSIC polynomial can be rewritten as

$$V(z_1,z_2)=\boldsymbol{c}(z_1,z_2)^{\mathrm{H}}\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}\boldsymbol{c}(z_1,z_2)=0 \tag{A1}$$

where

$$\boldsymbol{c}(z_1,z_2)=[\boldsymbol{a}_y(z_1)\otimes\boldsymbol{a}_x(z_2)]= [\boldsymbol{a}_x(z_2)^{\mathrm{T}},z_1\boldsymbol{a}_x(z_2)^{\mathrm{T}},\cdots,z_1^{N-1}\boldsymbol{a}_x(z_2)^{\mathrm{T}}]^{\mathrm{T}} \tag{A2}$$

To eliminate z_1^* and z_2^*, we can replace $\boldsymbol{c}(z_1,z_2)^{\mathrm{H}}$ with $z_1^{N-1}z_2^{M-1}\boldsymbol{c}(z_1^{-1},z_2^{-1})^{\mathrm{T}}$, then

$$V(z_1,z_2)=z_1^{N-1}z_2^{M-1}\boldsymbol{c}(z_1^{-1},z_2^{-1})^{\mathrm{T}}\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}\boldsymbol{c}(z_1,z_2)=0 \tag{A3}$$

Expand the coefficient of $V(z_1,z_2)$

$$V(z_1,z_2)=z_1^{N-1}z_2^{M-1}\boldsymbol{c}(z_1^{-1},z_2^{-1})^{\mathrm{T}}\boldsymbol{E}_{\mathrm{n}}\boldsymbol{E}_{\mathrm{n}}^{\mathrm{H}}\boldsymbol{c}(z_1,z_2)= a_1(z_2)z_1^{N-1}+a_2(z_2)z_1^{N-2}+a_3(z_2)z_1^{N-3}+\cdots+ a_{(N-1)}(z_2)z_1+a_N(z_2)+C \tag{A4}$$

where $a_1(z_2),\cdots,a_N(z_2)$ are all polynomials of z_2 and C is a constant.

According to Eq. (16), polynomial coefficients of $a_1(z_2),\cdots,a_N(z_2)$ are not all zero. Give z_2 an arbitrary value that satisfies

$$z_2=\mathrm{e}^{\mathrm{j}2\pi dv/\lambda}, v=\cos\theta\sin\phi\in[-1,1] \tag{A5}$$

then Eq.(A4) is converted into a univariate equation with only the variable z_1. The value of z_2 is an arbitrary complex number that satisfies the value range. There are infinitely many z_2 values and each z_2 corresponds to at least one z_1. According to Fundamental theorem of algebra, Eq.(11) contains infinitely many solutions in the complex number range.

References

[1] GODARA L C. Application of antenna arrays to mobile communications. II. Beam-forming and direction-of-arrival considerations[J]. Proceedings of the IEEE, 1997,85(8):1195-1245.

[2] KRIM H, VIBERG M. Two decades of array signal processing research: The parametric approach[J]. IEEE Signal Process Magazine,1996,13(4):67-94.

[3] ZHANG X, WANG F, XU D. Theory and application of array signal processing[M]. Beijing: National Defense Industry Press, 2013.

[4] ZHANG W, LIU W, WANG J, et al. Computationally efficient 2-D DOA estimation for uniform rectangular arrays[J]. Multidimensional Systems & Signal Processing,2014,25(4):847-857.

[5] CHEN Y M, LEE J H, YEH C C. Two-dimensional angle-of-arrival estimation for uniform planar arrays with sensor position errors[J]. IEE Proceedings F-Radar and Signal Processing,1993,140(1):37-42.

[6] HEIDENREICH P, RUBSAMEN M, RUBSAMEN M. Joint 2-D DOA estimation and phase calibration for uniform rectangular arrays[J]. IEEE Transactions on Signal Processing,2012,60(9):4683-4693.

[7] HUA Y, SARKAR T K, WEINER D D. An L-shaped array for estimating 2-D directions of wave arrival[J]. IEEE Transactions on Antennas & Propagation,1991,39(2):143-146.

[8] AL-JAZZAR S O, MCLERNON D C, SMADI M A. SVD-based joint azimuth/elevation estimation with automatic pairing[J]. Signal Processing,2010,90(5): 1669-1675.

[9] MATHEWS C P, ZOLTOWSKI M D. Eigen-structure techniques for 2-D angle estimation with uniform circular array[J]. IEEE Transactions on Signal Processing,1994,42(9):2395-2407.

[10] CHEN H, HOU C P, WANG Q, et al. Cumulants-based toeplitz matrices reconstruction method for 2-D coherent DOA estimation[J]. IEEE Sensors Journal, 2014,14(8):2824-2832.

[11] WAX M, SHAN T J, KAILATH T, et al. Spatio-temporal spectral analysis by eigenstructure methods[J]. IEEE Transactions on Acoustics, Speech and Signal Processing,1984,32(4):817-827.

[12] WU Q, SUN F, LAN P, et al. Two-dimensional direction-of-arrival estimation for co-prime planar arrays: A partial spectral search approach[J]. IEEE Sensors Journal,2016,16(14):5660-5670.

[13] WANG Y, CHEN J, FANG W. TST-MUSIC for joint DOA-delay estimation[J]. IEEE Transactions on Signal Processing,2001,49(4):721-729.

[14] ZHANG X, XU D. Angle estimation in MIMO radar using reduced-dimension Capon[J]. Electronics Letters, 2010, 46(12): 860-861.

[15] ZHANG X, ZHOU M, CHEN H, et al. Two-dimensional DOA estimation for acoustic vector-sensor array using a successive MUSIC[J]. Multidimensional Systems and Signal Processing, 2014, 25(3): 583-600.

[16] STOICA P, ARYE N. MUSIC, maximum likelihood, and Cramer-Rao bound[J]. IEEE Transactions on Acoustics, Speech and Signal Processing, 1989, 37(5): 720-741.

[17] ZHANG X, XU L, LEI X, et al. Direction of departure (DOD) and direction of arrival (DOA) estimation in MIMO radar with reduced-dimension MUSIC[J]. IEEE Communications Letters, 2010, 14(12): 1161-1163.

[18] ZHENG W, ZHANG X, SHI Z. Two-dimensional direction of arrival estimation for coprime planar arrays via a computationally efficient one-dimensional partial spectral search approach[J]. Iet Radar Sonar & Navigation, 2017, 11(10): 1581-1588.

[19] SAARNISAARI H. TLS-ESPRIT in a time delay estimation[C]//Processing of IEEE Vehicular Technology Conference. [S.l.]: IEEE, 1997, 3: 1619-1623.

[20] XU G, SILVERSTEIN S D, ROY R H, et al. Beamspace ESPRIT[J]. IEEE Transactions on Signal Processing, 1994, 42(2): 349-356.

[21] TAYEM N, KWON H M. L-shape 2-dimensional arrival angle estimation with propagator method[J]. IEEE Transactions on Antennas and Propagation, 2005, 53(5): 1622-1630.

[22] RAO B D, HARI K V S. Performance analysis of root-MUSIC[J]. IEEE Transactions on Acoustics, Speech and Signal Processing, 1989, 37(12): 1939-1949.

[23] ALI G, RAVIRAJ A. Low-complexity 2D root-MUSIC pairing for an L-shaped array[C]//Proceedings of 2017 IEEE Radar Conference. [S. l.]: IEEE, 2017: 957-962.

[24] BENCHEIKH M L, WANG Y, HE H, et al. Polynomial root finding technique for joint DOA DOD estimation in bistatic MIMO radar[J]. Signal Processing, 2010, 90(9): 2723-2730.

[25] ZHANG D, ZHANG Y, ZHENG G, et al. Two-dimensional direction of arrival estimation for coprime planar arrays via polynomial root finding technique[J]. IEEE Access, 2018, 6: 19540-19549.

[26] LEI H, TENG L, MAO E, et al. MMSE-based MDL method for robust estimation of number of sources without eigendecomposition[J]. IEEE Transactions on Signal Processing, 2009, 57(10): 4135-4142.

[27] WU H T, YANG J F, CHEN F K, et al. Source number estimator using Gerschgorin disks[C]//Proceedings of IEEE International Conference on Acoustics, Speech and Signal Processing. [S. l.]: IEEE, 1994: 261-264.

[28] STOICA P, NEHORAI A. Performance study of conditional and unconditional direction-of-arrival estimation[J]. IEEE Transactions on Signal Processing, 1990, 38(10): 1783-1795.

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均匀面阵中高效率的二维波达方向估计方法：降维求根MUSIC算法

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摘要：波达方向估计是阵列信号处理研究的重要方向之一。本文在降维求根技术和MUSIC算法的基础上，研究了均匀平面阵列的二维波达方向估计问题。首先基于噪声子空间和方向矢量之间的正交关系构造二维求根多项式，并证明该多项式包含无限多个解。为获取这些解中包含的真实目标参数，提出了一种新的低复杂度、计算效率高的算法，即降维求根MUSIC算法。所提算法应用降维求根技术目标的真实解，其中二维求根方程被转换为两次一维求根，该过程有效降低了求根难度。最后，通过一次配对过程获取目标角度参数的估计值。数值模拟验证了该方法的有效性和优越性。

关键词：均匀面阵；波达方向估计；降维求根MUSIC算法