Combined Gradient Representations for Generalized Birkhoffian Systems in Event Space and Its Stability Analysis

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Abstract: The combined gradient representations for generalized Birkhoffian systems in event space are studied. Firstly, the definitions of six kinds of combined gradient systems and corresponding differential equations are given. Secondly, the conditions under which generalized Birkhoffian systems become combined gradient systems are obtained. Finally, the characteristics of combined gradient systems are used to study the stability of generalized Birkhoffian systems in event space. Seven examples are given to illustrate the results.

Key words: generalized Birkhoffian system; event space; combined gradient systems; stability

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0 Introduction

In 1892, Lyapunov published his doctoral dissertation "general problems of motion stability", in which the concepts, research methods and related theories of stability were given. Since the end of the 19th century, Lyapunov stability theory is an important method to study the stability of mechanical system. Sometimes it is not so easy to construct Lyapunov function, and it is not very convenient in practical application[1-2]. In Ref.[3], two kinds of important systems were studied, in which one was gradient system, and the other is Hamilton system. Gradient system is a kind of mathematical system. The differential equation of the gradient equation is of first-order. Gradient system is especially suitable for studying the stability with Lyapunov function. If a mechanical system can be transformed into a gradient system, the properties of this gradient system can be used to study the behavior of the mechanical system, especially the stability problem[4-9]. Mei et al.[10] studied bifurcation for the generalized Birkhoffian system. Mei et al.[11-13] also studied skew-gradient representation and combined representation for the generalized Birkhoffian system. Two kinds of generalized gradient representations for generalized Birkhoffian systems were derived in Ref.[14]. The stability of the generalized Birkhoffian system was discussed by using the properties of the gradient system in Refs.[15-18]. Triple combined gradient systems representations for autonomous generalized Birkhoffian systems were given in Ref.[19]. A semi-negative definite matrix gradient system representation for non-autonomous generalized Birkhoffian system was given in Ref.[20]. Some progresses have been made on the gradient system representation and stability analysis for generalized Birkhoffian systems in configuration space. Event space is an extension of configuration space and time, and generalized coordinates are in the same position in event space, so parameters can be selected flexibly and relatively simple equations can be established. The parametric equation in event space can get not only the motion equation in the configuration space, but also the energy integral directly. There are few studies on generalized Birkhoffian dynamics in event space. Zhang[21] studied integrating factors and conservation laws of generalized Birkhoffian system dynamics in event space. Wu et al.[22] studied the gradient representation of holonomic system in event

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space. In this paper, six combined gradient system representations for generalized Birkhoffian systems in event space are discussed. Under certain conditions, a generalized Birkhoffian system in event space can become a combined gradient system, and then its stability can be discussed by using properties of the combined gradient system.

1 Generalized Birkhoff Equations in Event Space

In configuration space, we consider a generalized Birkhoffian system that is determined by $2n$ Birkhoff's variables a^μ ($\mu=1,2,\cdots,2n$). Now an event space of $(2n+1)$ dimensions is constructed. The coordinates of space points are t and a^μ. Introduce the notation

$$x_1=t \quad x_{\mu+1}=a^\mu \quad \mu=1,2,\cdots,2n \tag{1}$$

Then, all the variables x_α ($\alpha=1,2,\cdots,2n+1$) may be given as functions of some parameter τ. Let $x_\alpha=x_\alpha(\tau)$ be some curves of class C^2, such that

$$\frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau}=x'_\alpha \tag{2}$$

are not all zero at the same time, and

$$\dot{x}_\alpha=\frac{\mathrm{d}x_\alpha}{\mathrm{d}t}=\frac{x'_\alpha}{x'_1} \tag{3}$$

In configuration space, the Birkhoffian is $B=B(t,\boldsymbol{a})$, Birkhoff's functions are $R_\nu=R_\mu(t,\boldsymbol{a})$, and the additional terms are $\Lambda_\mu=\Lambda_\mu(t,\boldsymbol{a})$. In event space, Birkhoff's functions $B_\beta(x_\alpha)$($\beta=1,2,\cdots,2n+1$) are defined by the following equations[23].

$$\begin{cases}B_1(x_\alpha)=-B(x_1,x_2,\cdots,x_{2n+1})\\ B_{\mu+1}(x_\alpha)=R_\mu(x_1,x_2,\cdots,x_{2n+1}) \quad \mu=1,2,\cdots,2n\end{cases} \tag{4}$$

The additional terms are

$$\begin{cases}P_1=-x'_{\mu+1}\Lambda_\mu(x_1,x_2,\cdots,x_{2n+1})\\ P_{\mu+1}(x_\alpha)=x'_1\Lambda_\mu(x_1,x_2,\cdots,x_{2n+1})\end{cases}$$
$$\mu=1,2,\cdots,2n \tag{5}$$

In event space, the generalized Birkhoff equations are[23]

$$\left(\frac{\partial B_\beta}{\partial x_\alpha}-\frac{\partial B_\alpha}{\partial x_\beta}\right)x'_\beta=-P_\alpha \quad \alpha=1,2,\cdots,2n+1 \tag{6}$$

The generalized Birkhoff equations of Eq.(6) in event space are not independent each other, where the first equation of Eq.(6) is the result of the last $2n$ equations, and the last $2n$ equations can be written as

$$\left(\frac{\partial B_{\nu+1}}{\partial x_{\mu+1}}-\frac{\partial B_{\mu+1}}{\partial x_{\nu+1}}\right)x'_{\nu+1}+\left(\frac{\partial B_1}{\partial x_{\mu+1}}-\frac{\partial B_{\mu+1}}{\partial x_1}\right)x'_1=-P_{\mu+1} \quad \mu=1,2,\cdots,2n \tag{7}$$

Suppose that $x'_{\mu+1}$ can be solved from Eq.(6), that is

$$x'_{\mu+1}=\Omega^{\mu\nu}\left(\frac{\partial B_{\nu+1}}{\partial x_1}-\frac{\partial B_1}{\partial x_{\nu+1}}\right)x'_1-\Omega^{\mu\nu}P_{\nu+1}$$
$$\mu=1,2,\cdots,2n \tag{8}$$

where

$$\Omega_{\mu\nu}=\left(\frac{\partial B_{\mu+1}}{\partial x_{\nu+1}}-\frac{\partial B_{\nu+1}}{\partial x_{\mu+1}}\right) \quad \det(\Omega_{\mu\nu})\neq 0 \quad \Omega^{\mu\nu}\Omega_{\nu\rho}=\delta_{\mu\rho} \tag{9}$$

2 Combined Gradient System

2.1 Combined gradient system Ⅰ

This kind of combined gradient system is composed of generalized gradient system and generalized skew-gradient system. The differential equations of the system have the form as

$$\dot{x}_i=-\frac{\partial V(t,\boldsymbol{X})}{\partial x_i}+b_{ij}(X)\frac{\partial V(t,\boldsymbol{X})}{\partial x_j}$$
$$i,j=1,2,\cdots,m \tag{10}$$

where $(b_{ij}(\boldsymbol{X}))$ is the anti-symmetric matrix and $b_{ij}=-b_{ji}$. According to Eq.(10), we can obtain

$$\dot{V}=\frac{\partial V}{\partial t}-\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_i}+\frac{\partial V}{\partial x_i}b_{ij}\frac{\partial V}{\partial x_j}=\frac{\partial V}{\partial t}-\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_i} \tag{11}$$

where the second term at the right-hand side is less than zero. If V is positive definite, and $\dot{V}$ is negative definite, the solution of the system is asymptotically stable.

2.2 Combined gradient system Ⅱ

This kind of combined gradient system is composed of generalized gradient system and generalized gradient system with symmetric negative definite matrix. The differential equations of the system have the form as

$$\dot{x}_i=-\frac{\partial V(t,\boldsymbol{X})}{\partial x_i}+s_{ij}(\boldsymbol{X})\frac{\partial V(t,\boldsymbol{X})}{\partial x_j}$$
$$i,j=1,2,\cdots,m \tag{12}$$

where $(s_{ij}(\boldsymbol{X}))$ is symmetric negative definite matrix. According to Eq.(12), we can obtain

$$\dot{V}=\frac{\partial V}{\partial t}-\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_i}+\frac{\partial V}{\partial x_i}s_{ij}\frac{\partial V}{\partial x_j} \tag{13}$$

where the second and third terms at the right-hand side are less than zero. If V is positive definite, and

$\frac{\partial V}{\partial t} < 0$, the solution of the system is stable. If $\dot{V}$ is negative definite, the solution of the system is asymptotically stable.

2.3 Combined gradient system Ⅲ

This kind of combined gradient system is composed of generalized gradient system and generalized gradient system with negative semi-definite matrix. The differential equations have the form as

$$\dot{x}_i = -\frac{\partial V(t,\boldsymbol{X})}{\partial x_i} + a_{ij}(\boldsymbol{X})\frac{\partial V(t,\boldsymbol{X})}{\partial x_j}$$
$$i,j = 1,2,\cdots,m \tag{14}$$

where $(a_{ij}(\boldsymbol{X}))$ is the negative semi-definite matrix. According to Eq.(14), we can obtain

$$\dot{V} = \frac{\partial V}{\partial t} - \frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_i} + \frac{\partial V}{\partial x_i}a_{ij}\frac{\partial V}{\partial x_j} \tag{15}$$

where the second and third terms at the right-hand side are less than zero. If V is positive definite, and $\frac{\partial V}{\partial t} < 0$, the solution of the system is stable. If $\dot{V}$ is negative definite, the solution of the system is asymptotically stable.

2.4 Combined gradient system Ⅳ

This combined kind of gradient system is composed of generalized skew-gradient system and generalized gradient system with symmetric negative definite matrix. The differential equations have the form as

$$\dot{x}_i = b_{ij}(\boldsymbol{X})\frac{\partial V(t,\boldsymbol{X})}{\partial x_i} + s_{ij}(\boldsymbol{X})\frac{\partial V(t,\boldsymbol{X})}{\partial x_j}$$
$$i,j = 1,2,\cdots,m \tag{16}$$

where $(s_{ij}(\boldsymbol{X}))$ is the symmetric negative definite matrix. According to Eq.(16), we can obtain

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i}b_{ij}\frac{\partial V}{\partial x_j} + \frac{\partial V}{\partial x_i}s_{ij}\frac{\partial V}{\partial x_j} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i}s_{ij}\frac{\partial V}{\partial x_j} \tag{17}$$

where the second term at the right-hand side is less than zero. If V is positive definite, and $\dot{V}$ negative definite, the solution of the system is asymptotically stable.

2.5 Combined gradient system Ⅴ

This kind of combined gradient system is composed of generalized skew-gradient system and generalized gradient system with negative semi-definite matrix. The differential equations have the form

$$\dot{x}_i = b_{ij}(\boldsymbol{X})\frac{\partial V(t,\boldsymbol{X})}{\partial x_i} + a_{ij}(\boldsymbol{X})\frac{\partial V(t,\boldsymbol{X})}{\partial x_j}$$
$$i,j = 1,2,\cdots,m \tag{18}$$

where $(a_{ij}(\boldsymbol{X}))$ is the negative semi-definite matrix. According to Eq.(18), we can obtain

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i}b_{ij}\frac{\partial V}{\partial x_j} + \frac{\partial V}{\partial x_i}a_{ij}\frac{\partial V}{\partial x_j} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i}a_{ij}\frac{\partial V}{\partial x_j} \tag{19}$$

where the second term at the right-hand side is less than or equal to zero. If V is positive definite, and $\dot{V}$ negative definite, the solution of the system is asymptotically stable.

2.6 Combined gradient system Ⅵ

This kind of combined gradient system is composed by generalized gradient system with symmetric negative definite matrix and generalized gradient system with negative semi-definite matrix. The differential equations have the form as

$$\dot{x}_i = s_{ij}(\boldsymbol{X})\frac{\partial V(t,\boldsymbol{X})}{\partial x_i} + a_{ij}(\boldsymbol{X})\frac{\partial V(t,\boldsymbol{X})}{\partial x_j}$$
$$i,j = 1,2,\cdots,m \tag{20}$$

According to Eq.(20), we can obtain

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i}s_{ij}\frac{\partial V}{\partial x_j} + \frac{\partial V}{\partial x_i}a_{ij}\frac{\partial V}{\partial x_j} \tag{21}$$

where the second and third terms at the right-hand side are less than zero. If V is positive definite, and $\frac{\partial V}{\partial t} < 0$, the solution of the system is stable. If $\dot{V}$ is negative definite, the solution of the system is asymptotically stable.

3 Combined Gradient System Representations for Generalized Birkhoffian System in Event Space

In general, a generalized Birkhoffian system in event space is not a combined gradient system. For the system (8), if there are matrices $b_{\mu\nu}, s_{\mu\nu}, a_{\mu\nu}$ and the function V satisfies

$$\Omega^{\mu\nu}\left(\frac{\partial B_{\nu+1}}{\partial x_1} - \frac{\partial B_1}{\partial x_{\nu+1}}\right)x_1' - \Omega^{\mu\nu}P_{\nu+1} = -\frac{\partial V(\tau,a)}{\partial a^\mu} + b_{\mu\nu}\frac{\partial V(\tau,a)}{\partial a^\nu} \tag{22}$$

$$\Omega^{\mu\nu}\left(\frac{\partial B_{\nu+1}}{\partial x_1}-\frac{\partial B_1}{\partial x_{\nu+1}}\right)x_1'-\Omega^{\mu\nu}P_{\nu+1}=-\frac{\partial V(\tau,a)}{\partial a^\mu}+s_{\mu\nu}\frac{\partial V(\tau,a)}{\partial a^\nu}\tag{23}$$

$$\Omega^{\mu\nu}\left(\frac{\partial B_{\nu+1}}{\partial x_1}-\frac{\partial B_1}{\partial x_{\nu+1}}\right)x_1'-\Omega^{\mu\nu}P_{\nu+1}=-\frac{\partial V(\tau,a)}{\partial a^\mu}+a_{\mu\nu}\frac{\partial V(\tau,a)}{\partial a^\nu}\tag{24}$$

$$\Omega^{\mu\nu}\left(\frac{\partial B_{\nu+1}}{\partial x_1}-\frac{\partial B_1}{\partial x_{\nu+1}}\right)x_1'-\Omega^{\mu\nu}P_{\nu+1}=b_{\mu\nu}\frac{\partial V(\tau,a)}{\partial a^\nu}+s_{\mu\nu}\frac{\partial V(\tau,a)}{\partial a^\nu}\tag{25}$$

$$\Omega^{\mu\nu}\left(\frac{\partial B_{\nu+1}}{\partial x_1}-\frac{\partial B_1}{\partial x_{\nu+1}}\right)x_1'-\Omega^{\mu\nu}P_{\nu+1}=b_{\mu\nu}\frac{\partial V(\tau,a)}{\partial a^\nu}+a_{\mu\nu}\frac{\partial V(\tau,a)}{\partial a^\nu}\tag{26}$$

$$\Omega^{\mu\nu}\left(\frac{\partial B_{\nu+1}}{\partial x_1}-\frac{\partial B_1}{\partial x_{\nu+1}}\right)x_1'-\Omega^{\mu\nu}P_{\nu+1}=a_{\mu\nu}\frac{\partial V(\tau,a)}{\partial a^\nu}+s_{\mu\nu}\frac{\partial V(\tau,a)}{\partial a^\nu}\tag{27}$$

it can be transformed into combined gradient systems Ⅰ, Ⅱ, Ⅲ, Ⅳ, Ⅴ and Ⅵ, respectively.

4 Examples

Example 1 A generalized Birkhoffian system in event space is

$$\begin{cases}B_1=\frac{1}{2}x_2^2\left(1+\frac{1}{1+x_1}\right)+\frac{1}{2}x_3^2-x_2x_3\\ B_2=x_3\\ B_3=0\\ P_1=-x_3'x_2\left(2+\frac{1}{1+x_1}\right)\\ P_2=0\\ P_3=x_1'x_2\left(2+\frac{1}{1+x_1}\right)\end{cases}$$

We try to convert it into a combined gradient system and study the stability of its zero solution.

From Eq.(6), we obtain

$$\begin{cases}\left[x_3-x_2\left(1+\frac{1}{1+x_1}\right)\right]x_2'+(x_2-x_3)x_3'=x_3'x_2\left(2+\frac{1}{1+x_1}\right)\cdot\\ \left[x_2\left(1+\frac{1}{1+x_1}\right)-x_3\right]x_1'-x_3'=0\\ (x_3-x_2)x_1'+x_2'=-x_1'x_2\left(2+\frac{1}{1+x_1}\right)\end{cases}\tag{28}$$

From the last two equations, we have

$$\begin{cases}x_2'=-x_1'x_2\left(2+\frac{1}{1+x_1}\right)-(x_3-x_2)x_1'\\ x_3'=\left[x_2\left(1+\frac{1}{1+x_1}\right)-x_3\right]x_1'\end{cases}\tag{29}$$

Taking $\tau=x_1$, then $x_1'=1$ and Eq.(29) can be written as

$$\begin{cases}x_2'=-x_2\left(1+\frac{1}{1+\tau}\right)-x_3\\ x_3'=x_2\left(1+\frac{1}{1+\tau}\right)-x_3\end{cases}\tag{30}$$

Let $a^1=x_2, a^2=x_3$, then

$$\begin{pmatrix}(a^1)'\\(a^2)'\end{pmatrix}=\left(\begin{pmatrix}-1&0\\0&-1\end{pmatrix}+\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)\begin{pmatrix}\frac{\partial V}{\partial a^1}\\ \frac{\partial V}{\partial a^2}\end{pmatrix}\tag{31}$$

which is the combined gradient system Ⅰ, and the function V is

$$V=\frac{1}{2}(a^1)^2\left(1+\frac{1}{1+\tau}\right)+\frac{1}{2}(a^2)^2\tag{32}$$

V is positive definite and decreasing in the neighborhood of $a^1=a^2=0$. According to Eq.(32), we get

$$\dot V=-(a^1)^2\left[\left(1+\frac{1}{1+\tau}\right)^2+\frac{1}{2(1+\tau)^2}\right]-(a^2)^2$$

$\dot V$ is positive definite. The solution $a^1=a^2=0$ is asymptotically stable.

Example 2 A generalized Birkhoffian system in event space is

$$\begin{cases}B_1=-\frac{x_3^2}{2+\sin x_1}+x_2^2\\ B_2=x_3\\ B_3=0\\ P_1=x_2'\frac{6x_3^2}{2+\sin x_1}-4x_3'x_2\\ P_2=-x_1'\frac{6x_3^2}{2+\sin x_1}\\ P_3=4x_1'x_2\end{cases}$$

We try to convert it into a combined gradient system and study the stability of its zero solution.

From Eq.(6), we obtain

$$\begin{cases}(-2x_2)x_2'+\left(\dfrac{2x_3}{2+\sin x_1}\right)x_3'=-x_2'\dfrac{6x_3^2}{2+\sin x_1}+\\ \quad 4x_2x_3' \\ 2x_2x_1'-x_3'=x_1'\dfrac{6x_3^2}{2+\sin x_1} \\ -\dfrac{2x_3}{2+\sin x_1}x_1'+x_2'=-4x_1'x_2\end{cases} \tag{33}$$

From the last two equations, we have

$$\begin{cases}x_2'=\dfrac{2x_3}{2+\sin x_1}x_1'-4x_1'x_2 \\ x_3'=2x_2x_1'-x_1'\dfrac{6x_3^2}{2+\sin x_1}\end{cases} \tag{34}$$

Taking $\tau=x_1$, then $x_1'=1$ and Eq.(34) can be written as

$$\begin{cases}x_2'=\dfrac{2x_3}{2+\sin\tau}-4x_2 \\ x_3'=2x_2-\dfrac{6x_3^2}{2+\sin\tau}\end{cases} \tag{35}$$

Let $a^1=x_2$, $a^2=x_3$, then

$$\begin{pmatrix}(a^1)' \\ (a^2)'\end{pmatrix}=\left(\begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix}+\begin{pmatrix}-1 & 1 \\ 1 & -2\end{pmatrix}\right)\begin{pmatrix}\dfrac{\partial V}{\partial a^1} \\ \dfrac{\partial V}{\partial a^2}\end{pmatrix} \tag{36}$$

which is the combined gradient system Ⅱ, and the function V is

$$V=(a^1)^2+\frac{(a^2)^2}{2+\sin\tau} \tag{37}$$

V is positive definite and decreasing in the neighborhood of $a^1=a^2=0$. According to Eq.(37), we get

$$\dot{V}=-8(a^1)^2-(a^2)^2\frac{12+\cos\tau}{(2+\sin\tau)^2}+8\frac{a^1a^2}{2+\sin\tau}$$

$\dot{V}$ is positive definite. The solution $a^1=a^2=0$ is asymptotically stable.

Example 3 A generalized Birkhoffian system in event space is

$$\begin{cases}B_1=-x_3^2+4x_2x_3(2+\cos x_1)+x_2^2(2+\cos x_1) \\ B_2=x_3 \\ B_3=0 \\ P_1=4x_2'x_3(3+\cos x_1) \\ P_2=-4x_1'x_3(3+\cos x_1) \\ P_3=0\end{cases}$$

We try to convert it into a combined gradient system and study the stability of its zero solution.

From Eq.(6), we obtain

$$\begin{cases}(-4x_3-2x_2)(2+\cos x_1)x_2'+[2x_3-4x_2(2+\\ \quad \cos x_1)]x_3'=-4x_3x_2'(3+\cos x_1) \\ (4x_3+2x_2)(2+\cos x_1)x_1'-x_3'=4x_1'x_3(3+\\ \quad \cos x_1) \\ [-2x_3+4x_2(2+\cos x_1)]x_1'+x_2'=0\end{cases} \tag{38}$$

From the last two equations, we have

$$x_2'=[2x_3-4x_2(2+\cos x_1)]x_1'\ x_3'=(4x_3+2x_2)(2+\cos x_1)x_1'-4x_1'x_3(3+\cos x_1) \tag{39}$$

Taking $\tau=x_1$, then $x_1'=1$, Eq.(39) can be written as

$$\begin{cases}x_2'=2x_3-4x_2(2+\cos\tau) \\ x_3'=2x_2(2+\cos\tau)-4x_3\end{cases} \tag{40}$$

Let $a^1=x_2$, $a^2=x_3$, then

$$\begin{pmatrix}(a^1)' \\ (a^2)'\end{pmatrix}=\left(\begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix}+\begin{pmatrix}-1 & 1 \\ 1 & -1\end{pmatrix}\right)\begin{pmatrix}\dfrac{\partial V}{\partial a^1} \\ \dfrac{\partial V}{\partial a^2}\end{pmatrix} \tag{41}$$

which is the combined gradient system Ⅲ, and the function V is

$$V=(a^1)^2(2+\cos\tau)+(a^2)^2 \tag{42}$$

V is positive definite and decreasing in the neighborhood of $a^1=a^2=0$. Then, the solution $a^1=a^2=0$ is uniformity asymptotically stable.

Example 4 A generalized Birkhoffian system in event space is

$$\begin{cases}B_1=-4x_2x_3-2x_3^2 \\ B_2=x_3 \\ B_3=0 \\ P_1=-2x_3'x_2(4+\sin x_1) \\ P_2=0 \\ P_3=2x_1'x_2(4+\sin x_1)\end{cases}$$

We try to convert it into a combined gradient system and study the stability of its zero solution.

From Eq.(6), we obtain

$$\begin{cases}4x_3x_2'+(4x_3+4x_2)x_3'=2x_2x_3'(4+\sin x_1) \\ -4x_3x_1'-x_3'=0 \\ (-4x_3-4x_2)x_1'+x_2'=-2x_1'x_2(4+\sin x_1)\end{cases} \tag{43}$$

From the last two equations, we have

$$\begin{cases}x_2'=-2x_1'x_2(4+\sin x_1)+(4x_3+4x_2)x_1' \\ x_3'=-4x_3x_1'\end{cases} \tag{44}$$

Taking $\tau=x_1$, then $x_1'=1$ and Eq.(44) can be written as

$$\begin{cases}x_2'=-2x_2(2+\sin x_1)+4x_3 \\ x_3'=-4x_3\end{cases} \tag{45}$$

Let $a^1 = x_2, a^2 = x_3$, then

$$\begin{pmatrix}(a^1)'\\(a^2)'\end{pmatrix} = \left(\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} + \begin{pmatrix}-1 & 1\\1 & -2\end{pmatrix}\right)\begin{pmatrix}\dfrac{\partial V}{\partial a^1}\\ \dfrac{\partial V}{\partial a^2}\end{pmatrix} \tag{46}$$

which is the combined gradient system Ⅳ, and the function V is

$$V = (a^1)^2(2 + \sin\tau) + (a^2)^2 \tag{47}$$

V is positive definite and decreasing in the neighborhood of $a^1 = a^2 = 0$. Then, the solution $a^1 = a^2 = 0$ is uniformity asymptotically stable.

Example 5 A generalized Birkhoffian system in event space is

$$\begin{cases}B_1 = x_2^2[1 + \exp(-x_1)]\\ B_2 = x_3\\ B_3 = 0\\ P_1 = x_2' x_3[1 + \exp(-x_1)] - x_3' x_2[1 + \exp(-x_1)]\\ P_2 = -x_1' x_3[1 + \exp(-x_1)]\\ P_3 = x_1' x_2[1 + \exp(-x_1)]\end{cases}$$

We try to convert it into a combined gradient system and study the stability of its zero solution.

From Eq.(6), we obtain

$$\begin{cases}-2x_2[1+\exp(-x_1)]x_2' = (-x_3x_2' + x_2x_3')[1 + \\ \quad \exp(-x_1)]2x_2[1+\exp(-x_1)]x_1' - x_3' = \\ \quad x_1'x_3[1+\exp(-x_1)]\\ x_2' = -x_1'x_2[1+\exp(-x_1)]\end{cases} \tag{48}$$

From the last two equations, we have

$$\begin{cases}x_2' = -x_1'x_2[1+\exp(-x_1)]\\ x_3' = 2x_2[1+\exp(-x_1)]x_1' - x_1'x_3[1+\exp(-x_1)]\end{cases} \tag{49}$$

Taking $\tau = x_1$, then $x_1' = 1$, and Eq.(49) can be written as

$$\begin{cases}x_2' = -x_2[1 + \exp(-\tau)]\\ x_3' = (2x_2 - x_3)[1 + \exp(-\tau)]\end{cases} \tag{50}$$

Let $a^1 = x_2, a^2 = x_3$, then

$$\begin{pmatrix}(a^1)'\\(a^2)'\end{pmatrix} = \left(\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} + \begin{pmatrix}-1 & 1\\1 & -1\end{pmatrix}\right)\begin{pmatrix}\dfrac{\partial V}{\partial a^1}\\ \dfrac{\partial V}{\partial a^2}\end{pmatrix} \tag{51}$$

which is combined gradient system Ⅴ, and the function V is

$$V = \frac{1}{2}\left[(a^1)^2 + (a^2)^2\right][1 + \exp(-\tau)] \tag{52}$$

V is positive definite and decreasing in the neighborhood of $a^1 = a^2 = 0$. According to Eq.(52), we get

$$\dot{V} = -\left[(a^1)^2 + (a^2)^2\right]\left\{[1+\exp(-\tau)]^2 + \frac{1}{2}\exp(-\tau)\right\} + 2a^1a^2[1+exp(-\tau)]^2$$

$\dot{V}$ is positive definite. The solution $a^1 = a^2 = 0$ is asymptotically stable.

Example 6 A generalized Birkhoffian system in event space is

$$\begin{cases}B_1 = -x_3^2 + 4x_2x_3(2 + \sin x_1) + x_2^2(2 + \sin x_1)\\ B_2 = x_3\\ B_3 = 0\\ P_1 = 2x_2'x_3(7 + 2\sin x_1)\\ P_2 = -2x_1'x_3(7 + 2\sin x_1)\\ P_3 = 0\end{cases}$$

We try to convert it into a combined gradient system and study the stability of its zero solution.

From Eq.(6), we obtain

$$\begin{cases}(-4x_3 - 2x_2)(2 + \sin x_1)x_2' + [2x_3 - 4x_2(2 + \\ \quad \sin x_1)]x_3' = -2x_3x_2'(7 + 2\sin x_1)\\ (4x_3 + 2x_2)(2 + \sin x_1)x_1' - x_3' = 2x_1'x_3(7 + 2\sin x_1)\\ [-2x_3 + 4x_2(2 + \sin x_1)]x_1' + x_2' = 0\end{cases} \tag{53}$$

From the last two equations, we have

$$\begin{cases}x_2' = [2x_3 - 4x_2(2 + \sin x_1)]x_1'\\ x_3' = (4x_3 + 2x_2)(2 + \sin x_1)x_1' - 2x_1'x_3(7 + \\ \qquad 2\sin x_1)\end{cases} \tag{54}$$

Taking $\tau = x_1$, then $x_1' = 1$, and Eq.(54) can be written as

$$\begin{cases}x_2' = 2x_3 - 4x_2(2 + \sin x_1)\\ x_3' = 2x_2(2 + \sin x_1) - 6x_3\end{cases} \tag{55}$$

Let $a^1 = x_2, a^2 = x_3$, then

$$\begin{pmatrix}(a^1)'\\(a^2)'\end{pmatrix} = \left(\begin{pmatrix}-1 & 1\\1 & -1\end{pmatrix} + \begin{pmatrix}-1 & 0\\0 & -2\end{pmatrix}\right)\begin{pmatrix}\dfrac{\partial V}{\partial a^1}\\ \dfrac{\partial V}{\partial a^2}\end{pmatrix} \tag{56}$$

which is the combined gradient system Ⅵ, and the function V is

$$V = (a^1)^2(2 + \sin\tau) + (a^2)^2 \tag{57}$$

V is positive definite and decreasing in the neighborhood of $a^1 = a^2 = 0$. Then, the solution $a^1 = a^2 = 0$ is uniformity asymptotically stable.

Example 7 Linear damped oscillator

$$\ddot{x}+x-\gamma\dot{x}=0 \quad (\gamma=\text{const})$$

We try to convert it into a combined gradient system and study the stability of its zero solution in the event space.

In event space, the representation of generalized Birkhoffian system of the linear damped oscillator is

$$\begin{cases} B_1=-\dfrac{1}{2}\left[(x_2)^2+(x_3)^2-\gamma x_2x_3\right]\exp(-\gamma x_1) \\ B_2=\dfrac{1}{2}x_3\exp(-\gamma x_1) \\ B_3=-\dfrac{1}{2}x_2\exp(-\gamma x_1) \\ P_1=P_2=P_3=0 \end{cases}$$

From Eq.(6), we have

$$\begin{cases} -x_3'\exp(-\gamma x_1)-(x_2-\gamma x_3)\exp(-\gamma x_1)=0 \\ x_2'\exp(-\gamma x_1)-x_3\exp(-\gamma x_1)=0 \end{cases} \tag{58}$$

Let $a^1=x_2$, $a^2=x_3$, then

$$\begin{pmatrix}(a^1)' \\ (a^2)'\end{pmatrix}=\left(\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}+\begin{pmatrix}-1 & 1 \\ 1 & -1\end{pmatrix}\right)\begin{pmatrix}\dfrac{\partial V}{\partial a^1} \\ \dfrac{\partial V}{\partial a^2}\end{pmatrix} \tag{59}$$

which is the combined gradient system Ⅴ, and the function V is

$$V=\left[\frac{\gamma-2}{2}(a^2)^2-a^1a^2\right]\exp(-\gamma\tau) \tag{60}$$

When $\gamma\geqslant 6$, V is positive definite and decreasing in the neighborhood of $a^1=a^2=0$. The solution $a^1=a^2=0$ is uniformity asymptotically stable. When $\gamma<6$, V is not the Lyapunov function, then the stability of the solution is analyzed according to the characteristic roots of the linearized system.

5 Conclusions

It is an important and difficult problem to study the stability of constrained systems. The gradient of constrained mechanical systems is a new method for studying the stability of dynamic systems in analytical mechanics. In this paper, combined gradient systems are utilized to study the stability of generalized Birkhoffian systems in event space. If a generalized Birkhoffian system in event space satisfies the conditions (22—27), the generalized Birkhoffian system in event space can be transformed into a combined gradient system. Its dynamic behaviors can be discussed by using the properties of combined gradient systems, and some conclusions are given for the generalized Birkhoffian system in event space. Examples illustrate the application of the results.

References

[1] SANCHEZ D A. Ordinary differential equations and stability theory: An introduction[M]. New York: Dover Publications Inc, 2012.

[2] AGAFONOV S A. Stability and motion stabilization of non-conservative mechanical systems[J]. Journal of Mathematical Science,2002,112(5): 4419-4497.

[3] HIRSCH M W, SMALE S, DEVANEY R L. Differential equations, dynamical systems and an introduction to Chaos[M]. Singapore: Elsevier, 2013.

[4] SANTILLI R M. Foundation of theoretical mechanics Ⅰ[M]. New York: Springer-Verlag, 1978.

[5] SANTILLI R M. Foundation of theoretical mechanics Ⅱ[M]. New York: Springer-Verlag,1983.

[6] MCLACHLAN R I, QUISPEL G R W, ROBIDOUX N. Geometric integration using discrete gradients[J]. Philosophical Transactions of the Royal Society B Biological Sciences, 1998. DOI: 10.1098/rsta.1999.0363.

[7] BIRKHOFF G D. Dynamical systems[M]. Providence: AMS College Publisher,1927.

[8] MEI Fengxiang. Dynamics of generalized Birkhoffian systems[M]. Beijing: Science Press, 2013. (in Chinese)

[9] MEI Fengxiang, WU Huibin. Gradient representations of constrained mechanical systems[M]. Beijing: Science Press, 2016. (in Chinese)

[10] MEI F X, WU H B. Bifurcation for the generalized Birkhoffian system[J]. Chinese Physics B, 2015, 24(5): 419-420.

[11] MEI F X, WU H B. Skew-gradient representation of generalized Birkhoffian system[J]. Chinese Physics B, 2015, 24(10): 312-314.

[12] MEI F X, WU H B. Generalized Birkhoff system and a kind of combined gradient system[J]. Acta Physica Sinica, 2015, 64(18): 184501.

[13] MEI F X, CUI J C. Skew-gradient representations of constrained mechanical system[J]. Applied Mathematics and Mechanics, 2015, 36(7): 873-882.

[14] LI Y M, CHEN X W, WU H B, et al. Two kinds of generalized gradient representations for generalized Birkhoff systems[J]. Acta Physica Sinica, 2016, 65(8): 080201.

[15] CHEN X W, ZHANG Y, MEI F X. Stable general-

ized Birkhoff systems constructed by using a gradient system with non-symmetrical negative-definite matrix[J]. Chinese Journal of Theoretical and Applied Mechanics, 2017, 49(1): 149-153.

[16] CAO Qiupeng, CHEN Xiangwei. Stability and bifurcation for a type of non-autonomous generalized Birkhoffian system[J]. Acta Scientiarum Naturalium Universitatis Pekinesis, 2016, 52(4): 653-657. (in Chinese)

[17] CAO Qiupeng, ZHANG Yi, CHEN Xiangwei. Stability of equilibrium for autonomous generalized Birkhoffian system with constraints by gradient system[J]. Journal of Yunnan University (Natural Sciences Edition), 2015, 37(2): 228-232. (in Chinese)

[18] CHEN Xiangwei, CAO Qiupeng, MEI Fengxiang. Dependence of stability of generalized Birkhoff system on two parameters[J]. Chinese Quarterly of Mechanics, 2017, 38(1): 108-112. (in Chinese)

[19] WANG Jiahang, ZHANG Yi. Triple combined gradient system representations for autonomous generalized Birkhoffian system[J]. Journal of Yunnan University (Natural Sciences Edition), 2019, 41(3): 497-502. (in Chinese)

[20] WANG Jiahang, ZHANG Yi. A semi-negative definite matrix gradient system representation for non-autonomous generalized Birkhoff system[J]. Acta Scientiarum Naturalium Universitatis Sunyatseni, 2018, 57(3): 21-24. (in Chinese)

[21] ZHANG Y. Integrating factors and conservation laws of generalized Birkhoff system in event space[J]. Communications in Theoretical Physics, 2009, 51(6): 1078-1082.

[22] WU H B, MEI F X. A gradient representation of holonomic system in the event space[J]. Acta Physica Sinica, 2015, 64(23): 234501.

[23] ZHANG Y. Parametric equations and its first integrals for Birkhoffian systems in the event space[J]. Acta Physica Sinica, 2008, 57(5): 2649-2653.

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事件空间中广义Birkhoffian系统的组合梯度表示及其稳定性分析

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摘要:研究了事件空间中广义Birkhoffian系统的组合梯度系统表示。首先,给出了6类组合梯度系统的定义和微分方程;其次,得到了事件空间中广义Birkhoffian系统成为组合梯度系统的条件;最后,利用组合梯度系统的性质研究事件空间中广义Birkhoffian系统解的稳定性问题。举例说明结果的应用。

关键词:广义Birkhoffian系统;事件空间;组合梯度系统;稳定性