Generalized Canonical Transformations for Fractional Birkhoffian Systems

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Abstract: This paper presents fractional generalized canonical transformations for fractional Birkhoffian systems within Caputo derivatives. Firstly, based on fractional Pfaff-Birkhoff principle within Caputo derivatives, fractional Birkhoff's equations are derived and the basic identity of constructing generalized canonical transformations is proposed. Secondly, according to the fact that the generating functions contain new and old variables, four kinds of generating functions of the fractional Birkhoffian system are proposed, and four basic forms of fractional generalized canonical transformations are deduced. Then, fractional canonical transformations for fractional Hamiltonian system are given. Some interesting examples are finally listed.

Key words: fractional Birkhoffian system; generalized canonical transformation; fractional Pfaff-Birkhoff principle; generating function

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0 Introduction

As is known to all, the transformation of variables is an important means used by analytical mechanics to study problems. It is often very difficult to solve the general dynamical equation, so it is a very important research topic to use the method of variable transformation to make the differential equation to be easier to solve[1]. The transformation that keeps the form of Hamilton canonical equations unchanged is called canonical transformation. The purpose of canonical transformation is to find new Hamiltonian function through transformation, so that it has more concise forms and more cyclic coordinates, so as to simplify the solution of the problem. Hamilton canonical transformation is the basis of Hamilton-Jacobi equation and perturbation theory, and has a wide range of applications in celestial mechanics and other fields[2]. Under certain conditions, canonical transformation can be extended to nonholonomic systems[3-4] and weakly nonholonomic systems[5]. The transformation theory of Birkhoff's equations was first introduced by Santilli[6]. Wu and Mei[7] extended the transformation theory to the generalized Birkhoffian system. For Birkhoffian systems, we studied their generalized canonical transformations, and gave six kinds of transformation formulas[8-9]. The generalized canonical transformations were extended to second-order time-scale Birkhoffian systems[10].

In 1996, Riewe introduced fractional derivatives in his study of modeling of nonconservative mechanics[11]. In recent decades, fractional models have been widely used in various fields of mechanics and engineering due to their historical memory and spatial nonlocality, which can more succinctly and accurately describe complex dynamic behavior, material constitutive relations and physical properties[12-22]. However, the transformation theory based on fractional model is still an open subject. In Ref. [23], we presented fractional canonical transformations for fractional Hamiltonian systems. Here we will work on generalized canonical transformations

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of fractional Birkhoffian systems. We will set up the basic identity of constructing generalized canonical transformations. According to different cases of generating functions containing new and old variables, we will give four kinds of basic forms of generating functions and their corresponding generalized canonical transformation formulae.

1 Fractional Calculus

The fractional left derivative of Riemann-Liouville type is defined as[24]

$$ {}_{t_1}D_t^{\alpha}\xi(t)=\frac{1}{\Gamma(k-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k\int_{t_1}^{t}(t-\theta)^{k-\alpha-1}\xi(\theta)\mathrm{d}\theta \tag{1}$$

The right derivative is

$$ {}_tD_{t_2}^{\alpha}\xi(t)=\frac{1}{\Gamma(k-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^k\int_{t}^{t_2}(\theta-t)^{k-\alpha-1}\xi(\theta)\mathrm{d}\theta \tag{2}$$

The fractional left derivative of Caputo type is defined as

$$ {}_{t_1}^{\mathrm{C}}D_t^{\alpha}\xi(t)=\frac{1}{\Gamma(k-\alpha)}\int_{t_1}^{t}(t-\theta)^{k-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^k\xi(\theta)\mathrm{d}\theta \tag{3}$$

The right derivative is

$$ {}_t^{\mathrm{C}}D_{t_2}^{\alpha}\xi(t)=\frac{1}{\Gamma(k-\alpha)}\int_{t}^{t_2}(\theta-t)^{k-\alpha-1}\left(-\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^k\xi(\theta)\mathrm{d}\theta \tag{4}$$

where $\Gamma(y)=\int_0^{\infty}\mathrm{e}^{-x}x^{y-1}\mathrm{d}x$ is Gamma function and $k-1\leqslant\alpha<k$ the order of derivative. If α is an integer, then we have

$$\begin{cases} {}_{t_1}D_t^{\alpha}\xi(t)={}_{t_1}^{\mathrm{C}}D_t^{\alpha}\xi(t)=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha}\xi(t) \\ {}_tD_{t_2}^{\alpha}\xi(t)={}_t^{\mathrm{C}}D_{t_2}^{\alpha}\xi(t)=\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha}\xi(t) \end{cases} \tag{5}$$

The fractional-order integration by parts formulae are[15]

$$\int_{t_1}^{t_2}\eta(\theta)\,{}_{t_1}^{\mathrm{C}}D_{\theta}^{\alpha}\xi(\theta)\mathrm{d}\theta=\int_{t_1}^{t_2}\xi(\theta)\,{}_{\theta}D_{t_2}^{\alpha}\eta(\theta)\mathrm{d}\theta+\sum_{j=0}^{k-1}{}_{\theta}D_{t_2}^{\alpha+j-k}\eta(\theta)D^{k-1-j}\xi(\theta)\Big|_{t_1}^{t_2} \tag{6}$$

$$\int_{t_1}^{t_2}\eta(\theta)\,{}_{\theta}^{\mathrm{C}}D_{t_2}^{\alpha}\xi(\theta)\mathrm{d}\theta=\int_{t_1}^{t_2}\xi(\theta)\,{}_{t_1}D_{\theta}^{\alpha}\eta(\theta)\mathrm{d}\theta+\sum_{j=0}^{k-1}(-1)^{k+j}\,{}_{t_1}D_{\theta}^{\alpha+j-k}\eta(\theta)D^{k-1-j}\xi(\theta)\Big|_{t_1}^{t_2} \tag{7}$$

2 Fractional Birkhoffian Mechanics

The fractional Pfaff action can be written as

$$S=\int_{t_1}^{t_2}\left[R_{\beta}(t,a^{\gamma})\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\beta}-B(t,a^{\gamma})\right]\mathrm{d}t \tag{8}$$

where $R_{\beta}=R_{\beta}(t,a^{\gamma})$ $(\beta=1,2,\cdots,2n)$ are Birkhoff's functions, $B=B(t,a^{\gamma})$ is the Birkhoffian, and a^{γ} $(\gamma=1,2,\cdots,2n)$ are Birkhoff's variables.

The isochronous variational principle

$$\delta S=0 \tag{9}$$

with commutative relation

$$\delta\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\beta}={}_{t_1}^{\mathrm{C}}D_t^{\alpha}\delta a^{\beta} \tag{10}$$

and the endpoint condition

$$\delta a^{\beta}\big|_{t=t_1}=\delta a^{\beta}\big|_{t=t_2}=0 \tag{11}$$

is called the fractional Pfaff-Birkhoff principle within Caputo derivatives.

Expanding Principle (9) yields

$$\delta S=\int_{t_1}^{t_2}\left[{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\beta}\frac{\partial R_{\beta}}{\partial a^{\gamma}}\delta a^{\gamma}+R_{\beta}\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}\delta a^{\beta}-\frac{\partial B}{\partial a^{\beta}}\delta a^{\beta}\right]\mathrm{d}t=0 \tag{12}$$

Integrating by parts, and using Eqs.(6) and (11), we get

$$\int_{t_1}^{t_2}R_{\beta}\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}\delta a^{\beta}\mathrm{d}t=\int_{t_1}^{t_2}\delta a^{\beta}\,{}_tD_{t_2}^{\alpha}R_{\beta}\mathrm{d}t+\left(\delta a^{\beta}\,{}_tD_{t_2}^{\alpha-1}R_{\beta}\right)\Big|_{t_1}^{t_2}=\int_{t_1}^{t_2}\delta a^{\beta}\,{}_tD_{t_2}^{\alpha}R_{\beta}\mathrm{d}t \tag{13}$$

Substituting Eq.(13) into Eq.(12), we get

$$\int_{t_1}^{t_2}\left[\frac{\partial R_{\gamma}}{\partial a^{\beta}}\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\gamma}+{}_tD_{t_2}^{\alpha}R_{\beta}-\frac{\partial B}{\partial a^{\beta}}\right]\delta a^{\beta}\mathrm{d}t=0 \tag{14}$$

Since the interval $[t_1,t_2]$ is arbitrary, and δa^{β} is independent, we get

$$\frac{\partial R_{\gamma}}{\partial a^{\beta}}\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\gamma}+{}_tD_{t_2}^{\alpha}R_{\beta}-\frac{\partial B}{\partial a^{\beta}}=0\quad\beta=1,2,\cdots,2n \tag{15}$$

Eq.(15) can be called fractional Birkhoff's equations.

If take $\alpha\rightarrow1$, then Eq.(15) gives

$$\left(\frac{\partial R_{\gamma}}{\partial a^{\beta}}-\frac{\partial R_{\beta}}{\partial a^{\gamma}}\right)\dot{a}^{\gamma}-\frac{\partial B}{\partial a^{\beta}}-\frac{\partial R_{\beta}}{\partial t}=0 \tag{16}$$

Eq.(16) is Birkhoff's equation given in Ref.[6].

Let

$$\begin{cases} a^{\beta}=\begin{cases} q_{\beta} & \beta=1,2,\cdots,n \\ p_{\beta-n} & \beta=n+1,\cdots,2n \end{cases} \\ R_{\beta}=\begin{cases} p_{\beta} & \beta=1,2,\cdots,n \\ 0 & \beta=n+1,\cdots,2n \end{cases} \\ B=H \end{cases} \tag{17}$$

Then Principle (9) and Eq.(15) become

$$\delta S=\delta\int_{t_1}^{t_2}(p_{s\,t_1}^{\ C}D_t^{\alpha}q_s-H)\,\mathrm{d}t=0 \tag{18}$$

$$\begin{cases} {}_{t_1}^{C}D_t^{\alpha}q_s-\dfrac{\partial H}{\partial p_s}=0 \\ {}_tD_{t_2}^{\alpha}p_s-\dfrac{\partial H}{\partial q_s}=0 \end{cases} \quad s=1,2,\cdots,n \tag{19}$$

Eq.(18) is the fractional Hamilton principle and Eq.(19) is fractional Hamilton equations.

3 Fractional Generalized Canonical Transformations

The isochronous transformations from the old variable a^{β} to the new variable $\bar{a}^{\beta}$ are

$$t\to\bar{t}\equiv t,\ a^{\beta}\to\bar{a}^{\beta}(t,a^{\gamma}) \tag{20}$$

Let the transformed Birkhoffian and Birkhoff's functions be

$$\bar{B}=\bar{B}(t,\bar{a}^{\gamma}),\bar{R}_{\beta}=\bar{R}_{\beta}(t,\bar{a}^{\gamma})\quad \beta,\gamma=1,2,\cdots,2n \tag{21}$$

If Eq.(15) is still valid under the new variables $\bar{B}$ and $\bar{R}_{\beta}$, i.e.

$$\frac{\partial\bar{R}_{\gamma}}{\partial\bar{a}^{\beta}}\,{}_{t_1}^{C}D_t^{\alpha}\bar{a}^{\gamma}+{}_tD_{t_2}^{\alpha}\bar{R}_{\beta}-\frac{\partial\bar{B}}{\partial\bar{a}^{\beta}}=0 \tag{22}$$

then Eq. (20) is then called generalized canonical transformations of fractional Birkhoffian system (15). Obviously, if both the old and new variables satisfy

$$\delta\int_{t_1}^{t_2}(\bar{R}_{\beta\,t_1}^{\ \ C}D_t^{\alpha}\bar{a}^{\beta}-\bar{B})\,\mathrm{d}t=0 \tag{23}$$

$$\delta\int_{t_1}^{t_2}(R_{\beta\,t_1}^{\ \ C}D_t^{\alpha}a^{\beta}-B)\,\mathrm{d}t=0 \tag{24}$$

then Eq. (20) is the generalized canonical transformations. Since the starting and ending positions of the comparable motions of the system are defined, there are

$$\delta\int_{t_1}^{t_2}\frac{\mathrm{d}F}{\mathrm{d}t}\,\mathrm{d}t=\delta\Big[F(t)\Big|_{t_1}^{t_2}\Big]=0 \tag{25}$$

Based on Eqs.(23) and (24), considering Eq.(25), if the relationship between the old and new variables

$$R_{\beta}\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}a^{\beta})-\bar{R}_{\beta}\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^{\beta})+\big(\bar{B}-B\big)\mathrm{d}t=\mathrm{d}F(t,a^{\gamma},\bar{a}^{\gamma}) \tag{26}$$

is satisfied, the transformations are generalized canonical transformations of fractional Birkhoffian systems and vice versa. Eq.(26) is called the basic identity for constructing generalized canonical transformations. Because generalized canonical transformations depend entirely on the choice of any function F, it is called the generating function.

4 Generating Function and Transformations

For convenience, Birkhoff's variables are expressed as $\boldsymbol{a}=\{a^s,a_s\}$, and Birkhoff's functions are expressed as $\boldsymbol{R}=\{R_s,R^s\}$, where $s=1,2,\cdots,n$. Thus, Eq.(26) can be expressed as

$$R_s\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}a^s)+R^s\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}a_s)-\bar{R}_s\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^s)-\bar{R}^s\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_s)+\big(\bar{B}-B\big)\mathrm{d}t=\mathrm{d}F \tag{27}$$

where R_s and R^s are functions of t, a^j and a_j ($s,j=1,2,\cdots,n$), and $\bar{R}_s$ and $\bar{R}^s$ functions of t, $\bar{a}^j$ and $\bar{a}_j$. According to the fact that the generating function contains new and old variables, the following fractional generalized canonical transformations are presented.

4.1 Generalized canonical transformations based on generating functions of the first kind

Let the generating function be

$$F=F_1(t,{}_{t_1}^{C}D_t^{\alpha-1}a^s,{}_{t_1}^{C}D_t^{\alpha-1}a_s,{}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^s,{}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_s) \tag{28}$$

Then we have

$$\mathrm{d}F=\frac{\partial F_1}{\partial t}\mathrm{d}t+\frac{\partial F_1}{\partial\,{}_{t_1}^{C}D_t^{\alpha-1}a^s}\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}a^s)+\frac{\partial F_1}{\partial\,{}_{t_1}^{C}D_t^{\alpha-1}a_s}\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}a_s)+\frac{\partial F_1}{\partial\,{}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^s}\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^s)+\frac{\partial F_1}{\partial\,{}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_s}\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_s) \tag{29}$$

Substituting Eq.(29) into Eq.(27), we get

$$\left(\bar{B}-B-\frac{\partial F_1}{\partial t}\right)\mathrm{d}t+\left(R_s-\frac{\partial F_1}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s}\right)\cdot \mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s\right)+\left(R^s-\frac{\partial F_1}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s}\right)\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s\right)+\left(-\bar{R}_s-\frac{\partial F_1}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s}\right)\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s\right)+\left(-\bar{R}^s-\frac{\partial F_1}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s}\right)\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s\right)=0 \tag{30}$$

From Eq.(30), we have

$$\begin{cases} R_s=\dfrac{\partial F_1}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s},R^s=\dfrac{\partial F_1}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s} \\ \bar{R}_s=-\dfrac{\partial F_1}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s},\bar{R}^s=-\dfrac{\partial F_1}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s} \\ \bar{B}=B+\dfrac{\partial F_1}{\partial t} \end{cases} \tag{31}$$

4.2 Generalized canonical transformations based on generating functions of the second kind

Let the generating function be

$$F=F_2\left(t,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s,\bar{a}^s,\bar{a}_s\right)-\bar{R}_s\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s-\bar{R}^s\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s \tag{32}$$

Then we have

$$\mathrm{d}F=\frac{\partial F_2}{\partial t}\mathrm{d}t+\frac{\partial F_2}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s}\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s\right)+\frac{\partial F_2}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s}\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s\right)+\frac{\partial F_2}{\partial \bar{a}^s}\mathrm{d}\bar{a}^s+\frac{\partial F_2}{\partial \bar{a}_s}\mathrm{d}\bar{a}_s-\bar{R}_s\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s\right)-{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s\left(\frac{\partial \bar{R}_s}{\partial t}\mathrm{d}t+\frac{\partial \bar{R}_s}{\partial \bar{a}_j}\mathrm{d}\bar{a}_j+\frac{\partial \bar{R}_s}{\partial \bar{a}^j}\mathrm{d}\bar{a}^j\right)-\bar{R}^s\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s\right)-{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s\left(\frac{\partial \bar{R}^s}{\partial t}\mathrm{d}t+\frac{\partial \bar{R}^s}{\partial \bar{a}_j}\mathrm{d}\bar{a}_j+\frac{\partial \bar{R}^s}{\partial \bar{a}^j}\mathrm{d}\bar{a}^j\right) \tag{33}$$

Substituting Eq.(33) into Eq.(27), we get

$$\left(\bar{B}-B-\frac{\partial F_2}{\partial t}+\frac{\partial \bar{R}_j}{\partial t}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^j+\frac{\partial \bar{R}^j}{\partial t}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_j\right)\mathrm{d}t+\left(R_s-\frac{\partial F_2}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s}\right)\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s\right)+\left(R^s-\frac{\partial F_2}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s}\right)\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s\right)+\left(-\frac{\partial F_2}{\partial \bar{a}^s}+\frac{\partial \bar{R}_j}{\partial \bar{a}^s}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^j+\frac{\partial \bar{R}^j}{\partial \bar{a}^s}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_j\right)\mathrm{d}\bar{a}^s+\left(-\frac{\partial F_2}{\partial \bar{a}_s}+\frac{\partial \bar{R}_j}{\partial \bar{a}_s}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^j+\frac{\partial \bar{R}^j}{\partial \bar{a}_s}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_j\right)\mathrm{d}\bar{a}_s=0 \tag{34}$$

From Eq.(34), we have

$$\begin{cases} R_s-\dfrac{\partial F_2}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s}=0,R^s-\dfrac{\partial F_2}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s}=0 \\ -\dfrac{\partial F_2}{\partial \bar{a}^s}+\dfrac{\partial \bar{R}_j}{\partial \bar{a}^s}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^j+\dfrac{\partial \bar{R}^j}{\partial \bar{a}^s}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_j=0 \\ -\dfrac{\partial F_2}{\partial \bar{a}_s}+\dfrac{\partial \bar{R}_j}{\partial \bar{a}_s}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^j+\dfrac{\partial \bar{R}^j}{\partial \bar{a}_s}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_j=0 \\ \bar{B}-B-\dfrac{\partial F_2}{\partial t}+\dfrac{\partial \bar{R}_j}{\partial t}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^j+\dfrac{\partial \bar{R}^j}{\partial t}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_j=0 \end{cases} \tag{35}$$

4.3 Generalized canonical transformations based on generating functions of the third kind

Let the generating function be

$$F=F_3\left(t,a^s,a_s,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s\right)+R_s\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s+R^s\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s \tag{36}$$

Then we have

$$\mathrm{d}F=\frac{\partial F_3}{\partial t}\mathrm{d}t+\frac{\partial F_3}{\partial a^s}\mathrm{d}a^s+\frac{\partial F_3}{\partial a_s}\mathrm{d}a_s+\frac{\partial F_3}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s}\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s\right)+\frac{\partial F_3}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s}\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s\right)+R_s\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s\right)+{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s\left(\frac{\partial R_s}{\partial t}\mathrm{d}t+\frac{\partial R_s}{\partial a_j}\mathrm{d}a_j+\frac{\partial R_s}{\partial a^j}\mathrm{d}a^j\right)+R^s\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s\right)+{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s\left(\frac{\partial R^s}{\partial t}\mathrm{d}t+\frac{\partial R^s}{\partial a_j}\mathrm{d}a_j+\frac{\partial R^s}{\partial a^j}\mathrm{d}a^j\right) \tag{37}$$

Substituting Eq.(37) into Eq.(27), we get

$$\left(\bar{B}-B-\frac{\partial F_3}{\partial t}-\frac{\partial R_j}{\partial t}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^j-\frac{\partial R^j}{\partial t}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_j\right)\mathrm{d}t+\left(-\bar{R}_s-\frac{\partial F_3}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s}\right)\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s\right)+\left(-\bar{R}^s-\frac{\partial F_3}{\partial\, {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s}\right)\mathrm{d}\left({}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s\right)+\left(-\frac{\partial F_3}{\partial a^s}-\frac{\partial R_j}{\partial a^s}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^j-\frac{\partial R^j}{\partial a^s}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_j\right)\mathrm{d}a^s+\left(-\frac{\partial F_3}{\partial a_s}-\frac{\partial R_j}{\partial a_s}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^j-\frac{\partial R^j}{\partial a_s}{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_j\right)\mathrm{d}a_s=0 \tag{38}$$

From Eq.(38), we have

$$\begin{cases} -\bar{R}_s - \dfrac{\partial F_3}{\partial\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^s} = 0,\ -\bar{R}^s - \dfrac{\partial F_3}{\partial\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_s} = 0 \\ -\dfrac{\partial F_3}{\partial a^s} - \dfrac{\partial R_j}{\partial a^s}\, {}_{t_1}^{C}D_t^{\alpha-1}a^j - \dfrac{\partial R^j}{\partial a^s}\, {}_{t_1}^{C}D_t^{\alpha-1}a_j = 0 \\ -\dfrac{\partial F_3}{\partial a_s} - \dfrac{\partial R_j}{\partial a_s}\, {}_{t_1}^{C}D_t^{\alpha-1}a^j - \dfrac{\partial R^j}{\partial a_s}\, {}_{t_1}^{C}D_t^{\alpha-1}a_j = 0 \\ \bar{B} - B - \dfrac{\partial F_3}{\partial t} - \dfrac{\partial R_j}{\partial t}\, {}_{t_1}^{C}D_t^{\alpha-1}a^j - \dfrac{\partial R^j}{\partial t}\, {}_{t_1}^{C}D_t^{\alpha-1}a_j = 0 \end{cases} \tag{39}$$

4.4 Generalized canonical transformations based on generating functions of the fourth kind

Let the generating function be

$$F = F_4(t, a^s, a_s, \bar{a}^s, \bar{a}_s) + R_s\, {}_{t_1}^{C}D_t^{\alpha-1}a^s + R^s\, {}_{t_1}^{C}D_t^{\alpha-1}a_s - \bar{R}_s\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^s - \bar{R}^s\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_s \tag{40}$$

Then we have

$$\begin{aligned} \mathrm{d}F = {} & \frac{\partial F_4}{\partial t}\mathrm{d}t + \frac{\partial F_4}{\partial a^s}\mathrm{d}a^s + \frac{\partial F_4}{\partial a_s}\mathrm{d}a_s + \frac{\partial F_4}{\partial \bar{a}^s}\mathrm{d}\bar{a}^s + \\ & \frac{\partial F_4}{\partial \bar{a}_s}\mathrm{d}\bar{a}_s + R_s\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}a^s) + \\ & {}_{t_1}^{C}D_t^{\alpha-1}a^s\left(\frac{\partial R_s}{\partial t}\mathrm{d}t + \frac{\partial R_s}{\partial a_j}\mathrm{d}a_j + \frac{\partial R_s}{\partial a^j}\mathrm{d}a^j\right) + \\ & R^s\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}a_s) + \\ & {}_{t_1}^{C}D_t^{\alpha-1}a_s\left(\frac{\partial R^s}{\partial t}\mathrm{d}t + \frac{\partial R^s}{\partial a_j}\mathrm{d}a_j + \frac{\partial R^s}{\partial a^j}\mathrm{d}a^j\right) - \\ & \bar{R}_s\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^s) - \\ & {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^s\left(\frac{\partial \bar{R}_s}{\partial t}\mathrm{d}t + \frac{\partial \bar{R}_s}{\partial \bar{a}_j}\mathrm{d}\bar{a}_j + \frac{\partial \bar{R}_s}{\partial \bar{a}^j}\mathrm{d}\bar{a}^j\right) - \\ & \bar{R}^s\mathrm{d}({}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_s) - \\ & {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_s\left(\frac{\partial \bar{R}^s}{\partial t}\mathrm{d}t + \frac{\partial \bar{R}^s}{\partial \bar{a}_j}\mathrm{d}\bar{a}_j + \frac{\partial \bar{R}^s}{\partial \bar{a}^j}\mathrm{d}\bar{a}^j\right) \end{aligned} \tag{41}$$

Substituting Eq.(41) into Eq.(27), we get

$$\begin{aligned} & \left(\bar{B} - B - \frac{\partial F_4}{\partial t} - \frac{\partial R_j}{\partial t}\, {}_{t_1}^{C}D_t^{\alpha-1}a^j - \frac{\partial R^j}{\partial t}\, {}_{t_1}^{C}D_t^{\alpha-1}a_j - \frac{\partial \bar{R}_j}{\partial t}\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^j - \frac{\partial \bar{R}^j}{\partial t}\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_j\right)\mathrm{d}t + \\ & \left(-\frac{\partial F_4}{\partial a^s} - \frac{\partial R_j}{\partial a^s}\, {}_{t_1}^{C}D_t^{\alpha-1}a^j - \frac{\partial R^j}{\partial a^s}\, {}_{t_1}^{C}D_t^{\alpha-1}a_j\right)\mathrm{d}a^s + \\ & \left(-\frac{\partial F_4}{\partial a_s} - \frac{\partial R_j}{\partial a_s}\, {}_{t_1}^{C}D_t^{\alpha-1}a^j - \frac{\partial R^j}{\partial a_s}\, {}_{t_1}^{C}D_t^{\alpha-1}a_j\right)\mathrm{d}a_s + \\ & \left(-\frac{\partial F_4}{\partial \bar{a}^s} + \frac{\partial \bar{R}_j}{\partial \bar{a}^s}\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^j + \frac{\partial \bar{R}^j}{\partial \bar{a}^s}\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_j\right)\mathrm{d}\bar{a}^s + \\ & \left(-\frac{\partial F_4}{\partial \bar{a}_s} + \frac{\partial \bar{R}_j}{\partial \bar{a}_s}\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^j + \frac{\partial \bar{R}^j}{\partial \bar{a}_s}\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_j\right)\mathrm{d}\bar{a}_s = 0 \end{aligned} \tag{42}$$

From Eq.(42), we have

$$\begin{cases} -\dfrac{\partial F_4}{\partial a^s} - \dfrac{\partial R_j}{\partial a^s}\, {}_{t_1}^{C}D_t^{\alpha-1}a^j - \dfrac{\partial R^j}{\partial a^s}\, {}_{t_1}^{C}D_t^{\alpha-1}a_j = 0 \\ -\dfrac{\partial F_4}{\partial a_s} - \dfrac{\partial R_j}{\partial a_s}\, {}_{t_1}^{C}D_t^{\alpha-1}a^j - \dfrac{\partial R^j}{\partial a_s}\, {}_{t_1}^{C}D_t^{\alpha-1}a_j = 0 \\ -\dfrac{\partial F_4}{\partial \bar{a}^s} + \dfrac{\partial \bar{R}_j}{\partial \bar{a}^s}\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^j + \dfrac{\partial \bar{R}^j}{\partial \bar{a}^s}\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_j = 0 \\ -\dfrac{\partial F_4}{\partial \bar{a}_s} + \dfrac{\partial \bar{R}_j}{\partial \bar{a}_s}\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^j + \dfrac{\partial \bar{R}^j}{\partial \bar{a}_s}\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_j = 0 \\ \bar{B} - B - \dfrac{\partial F_4}{\partial t} - \dfrac{\partial R_j}{\partial t}\, {}_{t_1}^{C}D_t^{\alpha-1}a^j - \dfrac{\partial R^j}{\partial t}\, {}_{t_1}^{C}D_t^{\alpha-1}a_j - \\ \quad \dfrac{\partial \bar{R}_j}{\partial t}\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}^j - \dfrac{\partial \bar{R}^j}{\partial t}\, {}_{t_1}^{C}D_t^{\alpha-1}\bar{a}_j = 0 \end{cases} \tag{43}$$

It should be pointed out that the four fractional generalized canonical transformations determined by the four kinds of generating functions are only part of the transformations. Of course, only these four fractional generalized canonical transformations are quite extensive.

5 Canonical Transformations of Fractional Hamiltonian Systems

Let

$$a^s = q_s, a_s = p_s, R_s = p_s, R^s = 0, B = H \tag{44}$$

and

$$\bar{a}^s = \bar{q}_s, \bar{a}_s = \bar{p}_s, \bar{R}_s = \bar{p}_s, \bar{R}^s = 0, \bar{B} = \bar{H} \tag{45}$$

where q_s are the generalized coordinates, p_s the generalized momenta, and H is the Hamiltonian. Then Eq.(27) becomes

$$p_s\mathrm{d}\left({}_{t_1}^{C}D_t^{\alpha-1}q_s\right) - \bar{p}_s\mathrm{d}\left({}_{t_1}^{C}D_t^{\alpha-1}\bar{q}_s\right) + \left(\bar{H} - H\right)\mathrm{d}t = \mathrm{d}F \tag{46}$$

This is the basic identity for constructing canonical transformations of fractional Hamiltonian system. Thus, the results of generating functions and generalized canonical transformations of fractional Birkhoffian systems are reduced to generating functions and fractional canonical transformations of fractional Hamiltonian systems. The results are as follows:

(1) The first kind of generating function and corresponding fractional canonical transformation are

$$F=F_1\left(t,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}q_s,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{q}_s\right) \tag{47}$$

$$p_s=\frac{\partial F_1}{\partial\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}q_s},\bar{p}_s=-\frac{\partial F_1}{\partial\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{q}_s},\bar{H}=H+\frac{\partial F_1}{\partial t} \tag{48}$$

(2) The second kind of generating function and corresponding fractional canonical transformation are

$$F=F_2\left(t,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}q_s,\bar{p}_s\right)-\bar{p}_{s\,t_1}^{\ \,\mathrm{C}}D_t^{\alpha-1}\bar{q}_s \tag{49}$$

$$p_s=\frac{\partial F_2}{\partial\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}q_s},{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{q}_s=\frac{\partial F_2}{\partial\bar{p}_s},\bar{H}=H+\frac{\partial F_2}{\partial t} \tag{50}$$

(3) The third kind of generating function and corresponding fractional canonical transformation are

$$F=F_3\left(t,p_s,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{q}_s\right)+p_{s\,t_1}^{\ \,\mathrm{C}}D_t^{\alpha-1}q_s \tag{51}$$

$$\bar{p}_s=-\frac{\partial F_3}{\partial\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{q}_s},{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}q_s=-\frac{\partial F_3}{\partial p_s},\bar{H}=H+\frac{\partial F_3}{\partial t} \tag{52}$$

(4) The fourth kind of generating function and corresponding fractional canonical transformation are

$$F=F_4\left(t,p_s,\bar{p}_s\right)+p_{s\,t_1}^{\ \,\mathrm{C}}D_t^{\alpha-1}q_s-\bar{p}_{s\,t_1}^{\ \,\mathrm{C}}D_t^{\alpha-1}\bar{q}_s \tag{53}$$

$${}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}q_s=-\frac{\partial F_4}{\partial p_s},{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{q}_s=\frac{\partial F_4}{\partial\bar{p}_s},\ \bar{H}=H+\frac{\partial F_4}{\partial t} \tag{54}$$

When $\alpha\rightarrow1$, the results above are reduced to the classical integer-order generating functions and canonical transformations for Hamiltonian systems[1-2].

6 Examples

In the following, some simple but important examples are given to illustrate the effects of generating functions and fractional generalized canonical transformations.

Example 1 If the generating function F_1 is

$$F_1={}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s+{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s \tag{55}$$

then Eq.(31) gives

$$\begin{cases}R_s={}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s,R^s={}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s,\bar{R}_s=-{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s\\ \bar{R}^s=-{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s,\bar{B}=B\end{cases} \tag{56}$$

The transformation (56) shows that the new Birkhoff's functions $\bar{\boldsymbol{R}}=\left\{\bar{R}_s,\bar{R}^s\right\}$ depend on the old variables $\boldsymbol{a}=\{a^s,a_s\}$, and the old Birkhoff's functions $\boldsymbol{R}=\{R_s,R^s\}$ are associated with the new variables $\bar{\boldsymbol{a}}=\{\bar{a}^s,\bar{a}_s\}$.

Accordingly, for the fractional Hamiltonian system (19), let's take the generating function as

$$F_1={}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}q_s\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{q}_s \tag{57}$$

then the transformations are

$$p_s={}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{q}_s,\bar{p}_s=-{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}q_s,\bar{H}=H \tag{58}$$

Example 2 If the generating function F_2 is

$$F_2=\bar{R}_{s\,t_1}^{\ \,\mathrm{C}}D_t^{\alpha-1}a^s+\bar{R}^s\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s \tag{59}$$

then Eq.(35) gives

$$\begin{cases}R_s=\bar{R}_s,R^s=\bar{R}^s,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s={}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s\\ {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s={}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s,\bar{B}=B\end{cases} \tag{60}$$

Wherein, it is assumed that $\bar{\boldsymbol{R}}=\left\{\bar{R}_s,\bar{R}^s\right\}$ does not explicitly contain t. The transformations (60) show that the new Birkhoff's functions are the same as the old one, and the new Birkhoff's variables are the same as the old Birkhoff's ones, so the generating functioin (59) corresponds to the identity transformation.

Accordingly, for the fractional Hamiltonian system (19), let us take the generating function as

$$F_2=\bar{p}_{s\,t_1}^{\ \,\mathrm{C}}D_t^{\alpha-1}q_s \tag{61}$$

then the transformations are

$$p_s=\bar{p}_s,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{q}_s={}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}q_s,\bar{H}=H \tag{62}$$

This is an identity transformation.

Example 3 If the generating function F_3 is

$$F_3=R_{s\,t_1}^{\ \,\mathrm{C}}D_t^{\alpha-1}\bar{a}^s+R^s\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s \tag{63}$$

then Eq.(39) gives

$$\begin{cases}\bar{R}_s=-R_s,\bar{R}^s=-R^s,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s=-{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s\\ {}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s=-{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s,\bar{B}=B\end{cases} \tag{64}$$

Wherein, it is assumed that $\boldsymbol{R}=\{R_s,R^s\}$ does not explicitly contain t.

Accordingly, for the fractional Hamiltonian system (19), let us take the generating function as

$$F_3=p_{s\,t_1}^{\ \,\mathrm{C}}D_t^{\alpha-1}\bar{q}_s \tag{65}$$

then the transformations are

$$\bar{p}_s=-p_s,{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}q_s=-{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{q}_s,\bar{H}=H \tag{66}$$

Example 4 If the generating function F_4 is

$$F_4=R_s\bar{R}_s+R^s\bar{R}^s \tag{67}$$

then Eq.(43) gives

$$\begin{cases}\bar{R}_s=-{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a^s,\bar{R}^s=-{}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}a_s\\ R_s={}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}^s,R^s={}_{t_1}^{\mathrm{C}}D_t^{\alpha-1}\bar{a}_s,\bar{B}=B\end{cases} \tag{68}$$

Wherein, it is assumed that $\boldsymbol{R}=\{R_s,R^s\}$ and $\bar{\boldsymbol{R}}=\left\{\bar{R}_s,\bar{R}^s\right\}$ do not explicitly contain t. The transformations (68) are the same as transformations (56).

Therefore, the selection of different generating functions may correspond to the same generalized canonical transformations.

Accordingly, for the fractional Hamiltonian system (19), let us take the generating function as

$$F_4 = p_s \bar{p}_s \tag{69}$$

then the transformations are

$$\bar{p}_s = -{}_{t_1}^{C}D_t^{\alpha-1} q_s, p_s = {}_{t_1}^{C}D_t^{\alpha-1} \bar{q}_s,\ \bar{H} = H \tag{70}$$

7 Conclusions

In this paper, the generalized canonical transformations of fractional Birkhoffian systems are studied. Four basic forms of generalized canonical transformations are established by different choices of generating functions. The canonical transformations of fractional Hamiltonian systems are the special cases. As a novel mathematical tool, fractional calculus has been widely used in engineering, mechanics, materials and other research fields in recent years because it can more accurately describe complex dynamics problems with spatial nonlocality and historical memory. Birkhoffian mechanics is a new development of Hamiltonian mechanics, and canonical transformation is an important means of analytical mechanics, so the research on this topic is of great significance.

References

[1] MEI Fengxiang. Analytical mechanics (Ⅰ)(Ⅱ)[M]. Beijing: Beijing Institute of Technology Press, 2013. (in Chinese)

[2] CHEN Bin. Analytical dynamics[M]. Beijing: Peking University Press, 2012. (in Chinese)

[3] VAN DOOREN R. The generalized Hamilton-Jacobi method for nonholonomic dynamical system of Chetaev's type[J]. Journal of Applied Mathematics and Mechanics (ZAMM), 1975, 55(7/8): 407-411.

[4] VAN DOOREN R. Second form of the generalized Hamilton-Jacobi method for nonholonomic dynamical systems[J]. Journal of Applied Mathematics and Physics (ZAMP), 1978, 29(5): 828-834.

[5] MEI F X. Canonical transformation for weak nonholonomic systems[J]. Chinese Science Bulletin, 1993, 38 (4): 281-285.

[6] SANTILLI R M. Foundations of theoretical mechanics II[M]. New York: Springer-Verlag, 1983: 110-198

[7] WU Huibin, MEI Fengxiang. Transformation theory of generalized Birkhoffian systems[J]. Chinese Science Bulletin, 1995, 40(10): 885-888. (in Chinese)

[8] ZHANG Yi. The generalized canonical transformations of Birkhoffian systems and their basic formulations[J]. Chinese Quarterly of Mechanics, 2019, 40 (4): 656-665. (in Chinese)

[9] ZHANG Y. Theory of generalized canonical transformations for Birkhoff systems[J]. Advances in Mathematical Physics, 2020, 2020: 9482356.

[10] ZHANG Y, ZHAI X H. Generalized canonical transformation for second order Birkhoffian systems on time scales[J]. Theoretical and Applied Mechanics Letters, 2019, 9(6): 353-357.

[11] RIEWE F. Nonconservative Lagrangian and Hamiltonian mechanics[J]. Physical Review E, 1996, 53(2): 1890-1899.

[12] TARASOV V E. Fractional dynamics[M]. Beijing: Higher Education Press, 2010.

[13] ATANACKOVIĆ TM, KONJIK S, PILIPOVIĆ S, et al. Variational problems with fractional derivatives: Invariance conditions and Noether's theorem[J]. Nonlinear Analysis: Theory, Methods & Applications, 2009, 71(5/6): 1504-1517.

[14] HERRMANN R. Fractional calculus: An introduction for physicists[M]. Singapore: World Scientific Publishing, 2014.

[15] ALMEIDA R, POOSEH S, TORRES D F M. Computational methods in the fractional calculus of variations[M]. London: Imperial College Press, 2015.

[16] WANG Z H, HU H Y. Stability of a linear oscillator with damping force of the fractional-order derivative[J]. Science China: Physics, Mechanics & Astronomy, 2010, 53(2): 345-352.

[17] CHEN Wen, SUN Hongguang, LI Xicheng, et al. Fractional derivative modeling in mechanical and engineering problems[M]. Beijing: Science Press, 2010: 1-238. (in Chinese)

[18] ZHANG Y, ZHAI X H. Noether symmetries and conserved quantities for fractional Birkhoffian systems[J]. Nonlinear Dynamics, 2015, 81(1/2): 469-480.

[19] SONG C J, ZHANG Y. Conserved quantities and adiabatic invariants for fractional generalized Birkhoffian systems[J]. International Journal of Non-Linear Mechanics, 2017, 90: 32-38.

[20] ZHANG Y, LONG Z X. Fractional action-like variational problem and its Noether symmetries for a nonholonomic system[J]. Transactions of Nanjing University of Aeronautics and Astronautics, 2015, 32(4):

380-389.

[21] SONG C J, ZHANG Y. Discrete fractional Lagrange equations of nonconservative systems[J]. Transactions of Nanjing University of Aeronautics and Astronautics, 2019, 36(1): 175-180.

[22] SONG C J, ZHANG Y. Noether symmetry and conserved quantity for fractional Birkhoffian mechanics and its applications[J]. Fractional Calculus and Applied Analysis, 2018, 21(2): 509-526.

[23] ZHANG Yi. Theory of canonical transformation for a fractional mechanical system[J]. Acta Mathematicae Applicatae Sinica, 2016, 39(2): 249-260. (in Chinese)

[24] PODLUBNY I. Fractional differential equations[M]. San Diego: Academic Press, 1999.

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分数阶Birkhoff系统的广义正则变换

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摘要:研究给出Caputo导数下分数阶Birkhoff系统的分数阶广义正则变换。首先,基于Cuputo导数下分数阶Pfaff-Birkhoff原理,导出分数阶Birkhoff方程,建立构造广义正则变换的基本恒等式。其次,根据母函数含有新、旧变量的情况,提出分数阶Birkhoff系统的4类母函数,并导出相应的4种基本形式的分数阶广义正则变换。再次,给出分数阶Hamilton系统的分数阶正则变换。最后给出若干有趣的算例。

关键词:分数阶Birkhoff系统;广义正则变换;分数阶Pfaff-Birkhoff原理;母函数