A Preconditioned Fractional Tikhonov Regularization Method for Large Discrete Ill-posed Problems

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Abstract: The generalized Tikhonov regularization method is one of the most classical methods for the solution of linear systems of equations that arise from the discretization of linear ill-posed problems. However, the approximate solution obtained by the Tikhonov regularization method in general form may lack many details of the exact solution. Combining the fractional Tikhonov method with the preconditioned technique, and using the discrepancy principle for determining the regularization parameter, we present a preconditioned projected fractional Tikhonov regularization method for solving discrete ill-posed problems. Numerical experiments illustrate that the proposed algorithm has higher accuracy compared with the existing classical regularization methods.

Key words: fractional regularization; least-squares problem; regularization parameter

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0 Introduction

This paper aims to solve the linear system of equations

$$Ax = b \tag{1}$$

where the matrix $A \in \mathrm{R}^{n \times n}$ is severely ill-conditioned, and the right-hand side vector $b \in \mathrm{R}^n$ is determined through measurements and is contained by noise. Thus we have

$$b = b_{\text{exact}} + e$$

where $b_{\text{exact}} \in \mathrm{R}^n$ is the ideal output and e denotes the measurement error. Therefore, we prefer to compute the linear discrete ill-posed problem

$$A x_{\text{exact}} = b_{\text{exact}} \tag{2}$$

where x_{exact} denotes the ideal solution. To solve the problem (2), we replace the original problem (1) with the following least squares problem

$$\min_{x \in \mathrm{R}^n} \| Ax - b \| \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean norm. Then we would like to seek the solution of Eq.(3) to get an approximation of x_{exact}. Note that due to the tiny singular values of matrix A and the noise in b, the minimal Euclidean norm least squares solution of Eq.(3) is not a meaningful approximated solution of x_{exact}.

To avoid this deficiency, Tikhonov[1-2] replaces the original problem (1) with the following generalized regularized least squares problem

$$\min_{x \in \mathrm{R}^n} \{ \| Ax - b \|^2 + \mu^2 \| Lx \|^2 \} \tag{4}$$

where $\mu > 0$ is a regularization parameter to balances both terms for minimizing. L is a regularization matrix, and it is usually chosen as the identity matrix I with suitable size or the discrete first-order difference operator. Especially, when $L = I$, the problem (4) turned into standard form

$$\min_{x \in \mathrm{R}^n} \left\{ \| Ax - b \|^2 + \mu^2 \| x \|^2 \right\} \tag{5}$$

The problem (5) is equivalent to the following regularized linear system

$$(A^{\mathrm{T}} A + \mu^2 I) x = A^{\mathrm{T}} b \tag{6}$$

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with its solution

$$x_\mu=(A^{\mathrm{T}}A+\mu^2 I)^{-1}A^{\mathrm{T}}b \tag{7}$$

But the computed solution obtained by the Tikhonov method is too smooth and many details of the accurate solution will be lost. Therefore, in 2011, Hochstenbach et al.[3] proposed the fractional Tikhonov regularization method

$$\min_{x\in \mathrm{R}^n}\{\|Ax-b\|_W^2+\mu^2\|x\|^2\} \tag{8}$$

where $\|y\|_W=\left(y^{\mathrm{T}}Wy\right)^{1/2}$, and $W=(AA^{\mathrm{T}})^{(\alpha-1)/2}$ is symmetric positive semi-definite matrix. For the semi-norm $\|\cdot\|_W$, making $0<\alpha<1$ reduces over-smoothing. The different parameter α can be selected to improve the accuracy of the computed solution. When $\alpha=1$, it is the Tikhonov regularization in standard form.

To make the computed solution much closer to the exact solution, some scholars have applied the preconditioned technique to the original system for solving the preconditioned problem[4-5]. In Ref.[6], Gazzola constructed a new preconditioner based on the Arnoldi process, and solved the preconditioned problem by the Arnoldi-Tikhonov or the Arnoldi-TSVD (Tensor-singular value decomposition) method.

The main idea of this paper is to form a new regularization method by combining the projected fractional Tikhonov method with the preconditioned techniques in Ref.[6]. The remainder of this paper is organized as follows. Section 1 briefly reviews some of the previous work on preconditioner. Section 2 presents the projected fractional Tikhonov method at first, and then uses it to solve the preconditioned problem, which is the preconditioned projected fractional Tikhonov method mainly introduced in this paper. Section 3 explains how it can be used to compute the approximated solution. Numerical experiments and comparisons with other methods are given in Section 4, and some conclusions are given in Section 5.

1 Generation of Preconditioner

The generalized minimal residual (GMRES) is one of the most popular iterative methods for the solution of the linear discrete ill-posed problems based on the Arnoldi process, nevertheless, it is not very effective for solving the large-scale linear discrete ill-posed problems. In view of this situation, the Arnoldi decomposition was used in Ref.[6] to put forward the corresponding preconditioned technique to improve the deficiencies of GMRES[7].

The algorithm of Arnoldi process is given as follows.

Algorithm 1 The Arnoldi process

(1)Input: $x_0=0$, $A\in \mathrm{R}^n$, $b\in \mathrm{R}^n$;

(2)Compute: $r_0=A-bx_0$, $\beta=\|r_0\|$, $v_1=r_0/\|r_0\|$;

(3)for $k=1,2,\cdots,m$,

(4)Compute $w_j=Av_j$;

(5)for $i=1,2,\cdots,j$, do

(6)Compute $h_{ij}=(w_j,v_i)$;

(7)Compute $w_j=w_j-h_{ij}v_i$;

(8)Compute $h_{j+1,j}=\|w_j\|$;

(9)If $h_{j+1,j}=0$,

(10)Set $m=j$; end

(11)Else $v_{j+1}=w_j/h_{j+1,j}$;

(12)End for

(13)End for

Algorithm 1 generates orthonormal vectors $v_1, v_2, \cdots, v_m, v_{m+1}$, the first m of which form a basis for $K_m(A,b)=\mathrm{span}\{b, Ab, \cdots, A^{m-1}b\}$. Define the matrices $V_j=[v_1, v_2, \cdots, v_j]$ for $j\in\{m, m+1\}$. The scalars h_{ij} determined by the algorithm define an upper Hessenberg matrix $H_{m+1,m}=[h_{ij}]\in \mathrm{R}^{(m+1)\times m}$. So, using these matrices, the recursion formulas for Arnoldi process can be expressed as a partial Arnoldi decomposition

$$AV_m=V_{m+1}H_{m+1,m}$$

where m is the pre-determined dimension of the Krylov subspace to be projected. In Ref.[6], they terminated the initial Arnoldi process as soon as the following conditions hold , i.e.

$$h_{j+1,j}<\tau_1', \frac{|h_{j+1,j}-h_{j,j-1}|}{h_{j,j-1}}>\tau_1'' \tag{9}$$

where τ_1' and τ_1'' are certain user-specified parameters which are close to 0 and 1, respectively. Then the index j that meets Eq.(9) is the real Krylov subspace to be projected. Finally, four preconditioners M_1, M_2, M_3, M_4 are defined as

$$M_1=V_{j+1}H_{j+1,j}V_j^{\mathrm{T}}$$

$$M_2=V_jH_{j+1,j}^{\mathrm{T}}V_{j+1}^{\mathrm{T}}+(I-V_jV_j^{\mathrm{T}})$$

$$M_3 = V_{j+2} H_{j+2,j+1} V_{j+1}^{\mathrm{T}}$$
$$M_4 = V_j H_{j+2,j+1}^{\mathrm{T}} V_{j+2}^{\mathrm{T}} + (I - V_{j+1} V_{j+1}^{\mathrm{T}})$$

After being preconditioned, the original linear system $Ax = b$ is transformed into

$$\begin{cases} AMy = b \\ x = My \end{cases}$$

so the corresponding least squares problem is

$$\min_{y \in \mathbf{R}^n} \left\{ \| AMy - b \|^2 + \mu^2 \| y \|^2 \right\} \tag{10}$$

2 Preconditioned Projected Fractional Tikhonov Regularization Method

Considering the minimization problem (4), when the regularization matrix L takes the following finite difference operator

$$L_1 = \begin{bmatrix} 1 & -1 & & & 0 \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & -1 \end{bmatrix} \in \mathbf{R}^{(n-1)\times n} \tag{11}$$

$$L_2 = \begin{bmatrix} 1 & -2 & 1 & & & 0 \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & 1 & -2 & 1 \end{bmatrix} \in \mathbf{R}^{(n-2)\times n} \tag{12}$$

$$L_3 = \begin{bmatrix} -1 & 3 & -3 & 1 & & & 0 \\ & -1 & 3 & -3 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ 0 & & & -1 & 3 & -3 & 1 \end{bmatrix} \in \mathbf{R}^{(n-3)\times n} \tag{13}$$

Later, Morigis et al.[8] proposed an orthogonal projected operator as a regularization operator, which has the same null space as the finite difference operator. Define the orthogonal projected operator

$$L = I - PP^{\mathrm{T}},\ P \in \mathbf{R}^{l\times n},\ P^{\mathrm{T}} P = I \tag{14}$$

as a regularization operator, consequently the regularization operator defined by

$$P_1 = \frac{1}{\sqrt{n}} [1, 1, \cdots, 1]^{\mathrm{T}} \in \mathbf{R}^n \tag{15}$$

has the same null space as Eq. (11). Moreover, think about QR decomposition[9]

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & n \end{bmatrix} = P_2 R \tag{16}$$

where $P_2 \in \mathbf{R}^{n\times 2}$ is the orthogonal matrix, and R represents the upper triangular matrix. The regularization operator generated by P_2 has the same null space as the regularization operator (12). In fact, the matrix P can also be determined in many different ways, in which the regularization operators generated by some matrices can give computed solutions with higher accuracy. Then, define the weighted inverse of L as[10]

$$L_A^{\dagger} = (I - (A(I - L^{\dagger} L))^{\dagger} A) L^{\dagger} \in \mathbf{R}^{n\times k}$$

where $L^{\dagger} \in \mathbf{R}^{n\times k}$ represents the generalized inverse of the orthonormal projected regularization operator L. Let

$$\hat{A} = A L_A^{\dagger},\ \hat{x} = Lx, \hat{b} = b - Ax'$$
$$x' = (A(I - L^{\dagger} L))^{\dagger} b$$

The problem (4) can be converted to standard form

$$\min_{\hat{x} \in \mathbf{R}^n} \left\{ \| \hat{A}\hat{x} + \hat{b} \|^2 + \mu^2 \| \hat{x} \|^2 \right\} \tag{17}$$

In that way, the solution of Eq.(4) can also be obtained by the following equation

$$x_{\mu} = L_A^{\dagger} \hat{x} + x' \tag{18}$$

Furthermore, by using the orthogonal projected operator above, we will give a new regularization method. First, the general problem is converted to the standard form by using the orthogonal projected operator (15), and then the fractional power of the matrix is used as the weighted matrix to measure the residual error of the standard form (17). In other words, the minimum problem

$$\min_{\hat{x} \in \mathbf{R}^n} \left\{ \| \hat{A}\hat{x} + \hat{b} \|_W^2 + \mu^2 \| \hat{x} \|^2 \right\} \tag{19}$$

is used to replace the problem (17).

To solve the problem (19), it is transformed to solve the following normal equation

$$\left((\hat{A}\hat{A}^{\mathrm{T}})^{(\alpha+1)/2} + \mu^2 I \right) \hat{x} = (\hat{A}^{\mathrm{T}} \hat{A})^{(\alpha-1)/2} \hat{A}^{\mathrm{T}} \hat{b} \tag{20}$$

Consider the singular value decomposition of $\hat{A}$

$$\hat{A} = U \boldsymbol{\Sigma} V^{\mathrm{T}} \tag{21}$$

where U and V are unitary matrices[11], $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$, and $\sigma_1, \sigma_2, \cdots, \sigma_n$ are the singular values of $\hat{A}$. Substituting Eq.(21) into Eq.(20), the solution of Eq. (20) can be obtained as

$$\hat{x} = V (\boldsymbol{\Sigma}^{\alpha+1} + \mu^2 I)^{-1} \boldsymbol{\Sigma}^{\alpha} U^{\mathrm{T}} \hat{b}$$

That is

$$\hat{x}_{\mu,\alpha} = \sum_{i=1}^{n} \phi(\sigma_i)(u_i^{\mathrm{T}} \hat{b}) v_i \tag{22}$$

where

$$\phi(\sigma_i)=\frac{\sigma_i^{\alpha}}{\sigma_i^{\alpha+1}+\mu^2}$$

Therefore, by combining Eq.(22) with Eq.(18), the solution of the generalized minimum problem (4) can be obtained. The above is the projected fractional Tikhonov regularization method, denoted as the PFT algorithm.

Note that the method described on the above is based on the singular value decomposition of the coefficient matrix. However, for large-scale matrices, its singular value decomposition requires a very large amount of calculation, so it is necessary to first project the large-scale problem into the Krylov subspace with lower dimension and then solve the projected one. In 2009, Reichel et al.[12] proposed the Arnoldi-Tikhonov method based on this idea, which is briefly introduced below.

For large-scale matrices, the Arnoldi decomposition is performed to obtain

$$\boldsymbol{A}\boldsymbol{V}_m=\boldsymbol{V}_{m+1}\boldsymbol{H}_{m+1,m}$$

According to Algorithm 1, the matrix $\boldsymbol{V}_m$ satisfies

$$R(\boldsymbol{V}_m)=K_m(\boldsymbol{A},\boldsymbol{b})=\operatorname{span}\{\boldsymbol{b},\boldsymbol{A}\boldsymbol{b},\cdots,\boldsymbol{A}^{m-1}\boldsymbol{b}\}$$

In general $R(\boldsymbol{V}_m)\neq R(\boldsymbol{A})$, this decomposition is the basis of GMRES method. Assuming that the GMRES method is used to solve the linear equations (1) for m iterations, the computed solution is

$$\boldsymbol{x}_m=\boldsymbol{x}_0+V_m\boldsymbol{y}_m \qquad \boldsymbol{y}_m\in\mathrm{R}^m$$

When solving $\boldsymbol{x}_m\in K_m(\boldsymbol{A},\boldsymbol{b})$ through the Arnoldi process, it holds that

$$\begin{aligned}\|\boldsymbol{r}_m\|&=\min_{\boldsymbol{x}\in\boldsymbol{x}_0+K_m(\boldsymbol{A},\boldsymbol{b})}\|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}\|=\min_{\boldsymbol{y}\in R^m}\|\boldsymbol{A}\boldsymbol{V}_m\boldsymbol{y}-\boldsymbol{b}\|=\\&\min_{\boldsymbol{y}\in\mathrm{R}^m}\|\boldsymbol{V}_{m+1}\boldsymbol{H}_{m+1,m}\boldsymbol{V}_m\boldsymbol{y}-\boldsymbol{b}\|=\\&\min_{\boldsymbol{y}\in\mathrm{R}^m}\|\boldsymbol{V}_{m+1}(\boldsymbol{H}_{m+1,m}\boldsymbol{y}-\|\boldsymbol{b}\|\boldsymbol{e}_1)\|=\\&\min_{\boldsymbol{y}\in\mathrm{R}^m}\|\boldsymbol{H}_{m+1,m}\boldsymbol{y}-\|\boldsymbol{b}\|\boldsymbol{e}_1\|\end{aligned}\tag{23}$$

In fact, Eq.(23) shows that the original problem is transformed into the following minimization problem

$$\min_{\boldsymbol{y}\in\mathrm{R}^m}\|\boldsymbol{H}_{m+1,m}\boldsymbol{y}-\|\boldsymbol{b}\|\boldsymbol{e}_1\|\tag{24}$$

Then the Tikhonov method is used to solve the transformed problem, and the regularized solution is obtained. Above is the main idea of the Arnoldi-Tikhonov method. Combining the projected fractional Tikhonov (PFT) algorithm and the Arnoldi-Tikhonov method introduced on the above, a new algorithm Arnoldi-projected fractional Tikhonov regularization method is defined, which is recorded as the APFT algorithm.

The Arnoldi-projected fractional Tikhonov algorithm is given as follows.

Algorithm 2 The Arnoldi-projected fractional Tikhonov regularization process

(1) Input: $\boldsymbol{x}_0=0$, $A\in\mathrm{R}^n$, $\boldsymbol{b}\in\mathrm{R}^n$;

(2) Compute: $\boldsymbol{r}_0=\boldsymbol{A}-\boldsymbol{b}\boldsymbol{x}_0$, $\boldsymbol{\beta}=\|\boldsymbol{r}_0\|$, $\boldsymbol{v}_1=\boldsymbol{r}_0/\|\boldsymbol{r}_0\|$;

(3) for $k=1,2,\cdots,m$

(4) Compute $\boldsymbol{w}_j=\boldsymbol{A}\boldsymbol{v}_j$;

(5) for $i=1,2,\cdots,j$, do

(6) Compute $\boldsymbol{h}_{ij}=(\boldsymbol{w}_j,\boldsymbol{v}_i)$;

(7) Compute $\boldsymbol{w}_j=\boldsymbol{w}_j-\boldsymbol{h}_{ij}\boldsymbol{v}_i$;

(8) Compute $\boldsymbol{h}_{j+1,j}=\|\boldsymbol{w}_j\|$;

(9) If $\boldsymbol{h}_{j+1,j}=0$,

(10) Set $m=j$; end

(11) Else $\boldsymbol{v}_{j+1}=\boldsymbol{w}_j/\boldsymbol{h}_{j+1,j}$;

(12) End for

(13) End for

(14) The PFT algorithm is used to solve the minimization problem (24), and the computed solution is denoted as $\boldsymbol{y}_m$.

(15) Restore regularized solution: $\boldsymbol{x}_m=\boldsymbol{x}_0+\boldsymbol{V}_m\boldsymbol{y}_m$.

3 Solution to Preconditioned Problem

The section discusses the application of Algorithm 2 and the Arnoldi-Tikhonov algorithm to solve the preconditioned system (10), so the problem (10) is projected into the Krylov subspace $\boldsymbol{V}_k$, where

$$\boldsymbol{A}\boldsymbol{M}\boldsymbol{V}_k=\boldsymbol{V}_{k+1}\boldsymbol{H}_{k+1,k}\tag{25}$$

and $\boldsymbol{k}$ is actually*j*selected by satisfying the inequality (9) in the second section. The minimization problem (10) is transformed by Eq.(25) into[6]

$$\min_{\boldsymbol{z}\in\mathrm{R}^k}\left\{\|\boldsymbol{H}_{k+1,k}\boldsymbol{z}-\|\boldsymbol{b}\|\boldsymbol{e}_1\|+\mu^2\|\boldsymbol{L}\boldsymbol{z}\|^2\right\}\tag{26}$$

Next, the PFT algorithm is used to obtain the computed solution (26) of the problem as $\boldsymbol{z}_\mu$, so that the solution of the problem (10) is $\boldsymbol{y}_\mu=\boldsymbol{V}_k\boldsymbol{z}_\mu$ which gives the computed solution $\boldsymbol{x}_\mu=\boldsymbol{M}\boldsymbol{y}_\mu$ of the problem (4). This preconditioned Arnoldi-projected fractional Tikhonov method is denoted as P-PFT algorithm. In particular, when using the P-PFT algorithm, we choose μ that satisfies the discrepancy principle

$$\| H_{k+1,k} z - \| b \| e_1 \| \leqslant \tau\delta$$

where δ is the bound for the noise in $\boldsymbol{b}$, and let τ be a user-specified value that independent of $\boldsymbol{\delta}$.

4 Numerical Experiments

In this section, we will construct some experiments to illustrate the effectiveness of the P-PFT algorithm for solving linear discrete ill-posed problems. All the computations were carried out in MATLAB R2016b on personal computer with 1.86 GHz Intel Core i4, 4 GB DDR3. For all the tests, the initial guess $\boldsymbol{x}_0 = \boldsymbol{0}$ and the stopping criteria (9) are used with $\tau_1' = 0.9$, $\tau_1'' = 10^{-4}$. Besides, the maximum allowed number of the Arnoldi iterations in algorithm 1 is $m = 200$. We use the relative error norm, defined by $\| \boldsymbol{x} - \boldsymbol{x}_{\text{exact}} \| / \| \boldsymbol{x}_{\text{exact}} \|$, as a measure of the accuracy of the computed solution.

4.1 Example 1

Consider the linear systems $\boldsymbol{Ax} = \boldsymbol{b}$, where $A \in \mathbb{R}^{3\,000 \times 3\,000}$ is the foxgood matrix[13], and the right-hand side vector $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_{\text{exact}} + \delta \cdot \text{rand}(3\,000, 1)$ with relative noise level

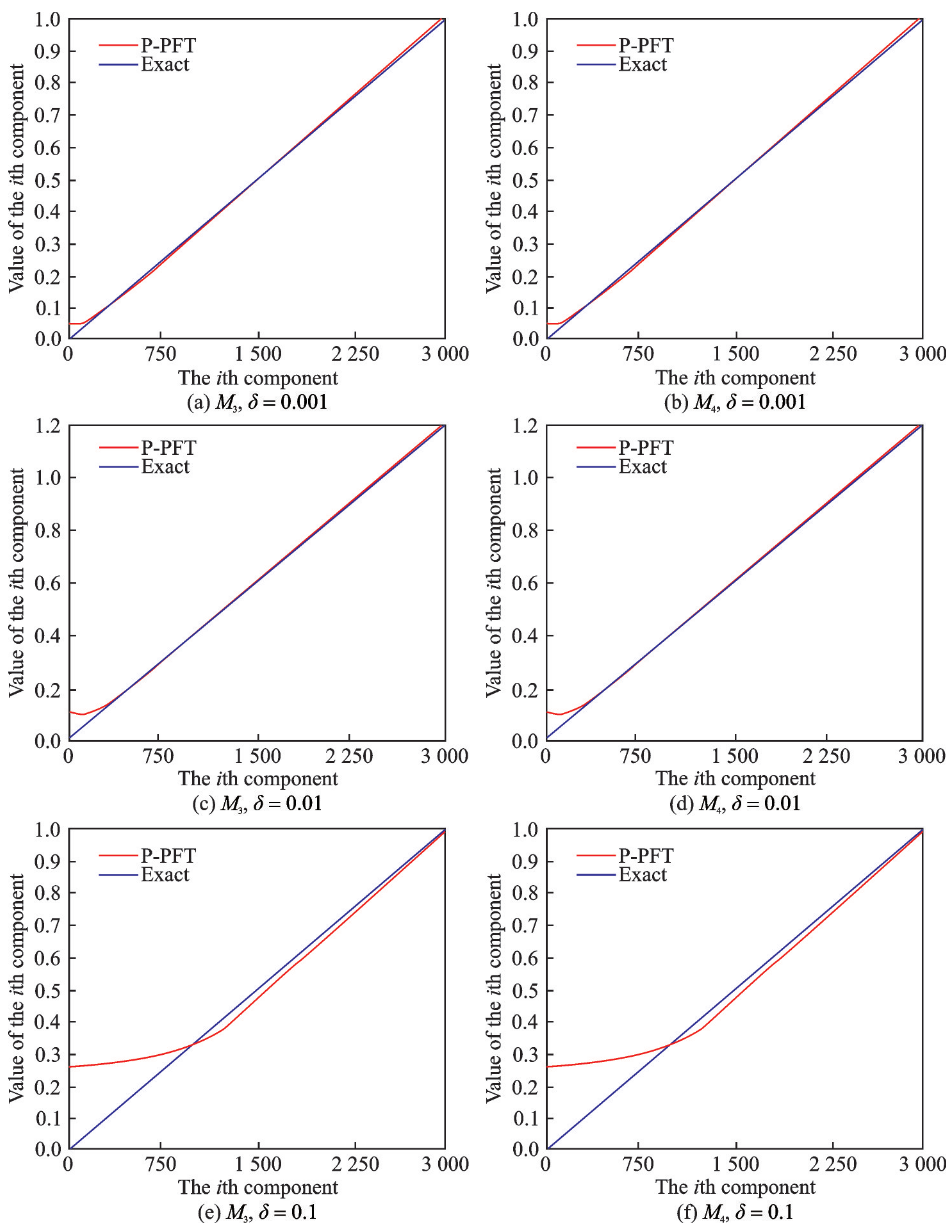

Fig.1 Exact solution of the foxgood problem and the computed solution obtained by P-PFT algorithm

$\delta = 0.001, 0.01, 0.1$. Let the orthogonal projected operator be generated by Eq.(16) and the regularization parameter $\alpha = 0.1$. The numerical results of applying P-PFT algorithm equipped with preconditioner $\boldsymbol{M}_3, \boldsymbol{M}_4$ are shown in Fig.1 that illustrates the distance between the computed solution and the exact solution. It can be seen from the Fig.1 that the computed solution obtained by using the P-PFT algorithm can approximate the exact solution well, which indicates that our proposed method is effective.

4.2 Example 2

Consider the linear systems $\boldsymbol{Ax} = \boldsymbol{b}$, where $A \in \mathrm{R}^{3000 \times 3000}$ takes the foxgood, baart matrix[13], and the right-hand side vector $\boldsymbol{b} = \boldsymbol{Ax}_{\text{exact}} + \delta \cdot \text{rand}(3\,000, 1)$ with relative noise level $\delta = 0.001, 0.01, 0.1$. Let the orthogonal projected operator be generated by Eq.(16) and the regularization parameter $\alpha = 0.1$. Then compare the relative error of the computed solution obtained by the P-PFT algorithm with different preconditioners. The numerical results of comparing the relative error that applying P-PFT algorithm and P-AT algorithm equipped with preconditioners $\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{M}_3, \boldsymbol{M}_4$ are shown in Tables 1,2.

Table 1 Comparison of the relative error for P-PFT algorithm and P-AT algorithm in solving baart problem

Baart	$\delta=0.001$		$\delta=0.01$		$\delta=0.1$	
	P-AT	P-PFT	P-AT	P-PFT	P-AT	P-PFT
$\boldsymbol{M}_1$	0.171	0.138	0.201	0.201	0.413	0.337
$\boldsymbol{M}_2$	0.176	0.140	0.199	0.183	0.413	0.338
$\boldsymbol{M}_3$	0.171	0.143	0.200	0.171	0.415	0.329
$\boldsymbol{M}_4$	0.170	0.143	0.200	0.173	0.410	0.320

Table 2 Comparison of the relative error for P-PFT algorithm and P-AT algorithm in solving foxgood problem

Foxgood	$\delta=0.001$		$\delta=0.01$		$\delta=0.1$	
	P-AT	P-PFT	P-AT	P-PFT	P-AT	P-PFT
$\boldsymbol{M}_1$	0.015 6	0.008 0	0.038 8	0.028 6	0.044 7	0.049 5
$\boldsymbol{M}_2$	0.015 7	0.008 1	0.038 7	0.029 1	0.045 0	0.064 2
$\boldsymbol{M}_3$	0.016 0	0.008 5	0.038 6	0.028 4	0.045 1	0.049 8
$\boldsymbol{M}_4$	0.017 0	0.008 6	0.040 0	0.028 7	0.044 9	0.049 7

Table 1 shows that the P-PFT algorithm can give the more accuracy computed solution than the P-AT algorithm for the test problem baart under all noise levels. Moreover, the approximate solution computed when using the preconditioner $\boldsymbol{M}_4$ does look much improved for the test problem foxgood in Table 2. It can also be found from Table 2 that the P-PFT algorithm performs much better than the P-AT algorithm for the test problem foxgood when the relative noise is smaller.

5 Conclusions

In this paper, a new regularization method is formed by combining the preconditioned technique for solving discrete ill-posed problems with the projected Tikhonov method based on the projected operator. The numerical experiments in this paper show that the preconditioned Arnoldi-projected fractional Tikhonov method has the advantages of more accurate results, so the method is effective and feasible to some extent.

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求解大型离散不适定问题的预处理分数阶Tikhonov正则化方法

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摘要:广义Tikhonov正则化方法是求解线性离散不适定问题的经典方法之一,但标准形式的Tikhonov正则化方法所求近似解缺乏精确解的许多细节。本文将分数阶Tikhonov方法与预条件技术相结合,利用偏差原理确定正则化参数,提出了求解大型离散不适定问题的预处理分数阶Tikhonov正则化方法。数值实验表明,与现有的经典正则化方法相比,该算法具有更高的精确度。

关键词:分数阶正则化;最小二乘问题;正则化参数