Perturbation to Noether Symmetries and Adiabatic Invariants for Generalized Birkhoff Systems Based on El-Nabulsi Dynamical Model

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Abstract: With the action of small perturbation on generalized El-Nabulsi-Birkhoff fractional equations, the perturbation to Noether symmetries and adiabatic invariants are studied under the framework of El-Nabulsi′s fractional model. Firstly, based on the invariance of El-Nabulsi-Pfaff action under the infinitesimal transformations of group, the exact invariants are given. Secondly, on the basis of the definition of higher order adiabatic invariants of a dynamical system, the adiabatic invariants of the Noether symmetric perturbation for disturbed generalized El-Nabulsi′s fractional Birkhoff system are presented under some conditions, and some special cases are discussed. Finally, an example known as Hojman-Urrutia problem is given to illustrate the application of the results.

Key words: perturbation to Noether symmetry; adiabatic invariant; El-Nabulsi dynamical model; generalized Birkhoff system; infinitesimal transformation

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0 Introduction

In 1927, a new integral variational principle was introduced by an American mathematician—Birkhoff, and a new form of the equations of motion was also obtained in his monograph[1]. From then on, Birkhoffian dynamics gained significant headways. For instance, in 1983, the Birkhoff equations and the transformation theory of Birkhoff equations are studied by Santilli[2]; In 1996, the theoretical framework of Birkhoffian dynamics was established by Mei and his co-workers[3] (They extended the Birkhoff system to a generalized Birkhoffian system, and obtained a series of results[4-6].); In 1997, the symmetry of the Birkhoffian system is presented[7], to name just a few.

Fractional calculus can be used to investigate complex dynamical systems and understand complicated physical processes. And based on the fractional calculus, Riewe[8-9] studied the fractional variational problems, and established the fractional Euler-Lagrange equations as well as the fractional Hamilton equations. Since then, many further researches on fractional variational problems have been found[10-18]. El-Nabulsi′s fractional model, a fractional action-like variational approach based on the fractional calculus, was introduced by El-Nabulsi[19] in 2005 when he was studying nonconservative dynamical modeling. Subsequently, this method was widely used and many results have been obtained. For instance, El-Nabulsi generalized the approach to a Lagrangian which depends on Riemann-Liouville fractional derivatives[20], to periodic functional or exponential law[21-22], and to multi-dimentional frac-

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tional action-like variational problems[23]. Apart from these results, in 2011, El-Nabulsi[24] gave the universal fractional action-like Euler-Lagrange equations on the basis of a generalized fractional derivative operator and, Herzallah et al[25] presented the fractional action-like Hamilton-Jacobi theory. In 2013, Zhang and Zhou[26] introduced the idea of El-Nabulsi's fractional model to Birkhoffian mechanics, on the basis of El-Nabulsi's fractional model. They first presented the fractional Pfaff variational problem, established the El-Nabulsi-Birkhoff fractional equations of motion, with which the Birkhoff system is called the El-Nabulsi's fractional Birkhoff system, and obtained the El-Nabulsi's fractional Noether theorems. Moreover, in 2014, Zhang and Ding[27] presented the generalized El-Nabulsi-Birkhoff fractional equations and the generalized El-Nabulsi's fractional Birkhoff system, and established the El-Nabulsi's fractional Noether theorems.

Perturbation to symmetry and adiabatic invariants for a dynamical system are of great significance in many fields, such as mechanics, mathematics and physics. Adiabatic invariant was first proposed by Burgers in 1917[28]. For a mechanical system, the relation existing in the integrability and the variations of its symmetries and invariants under the action of small disturbance is so intimate that the researches on perturbation to symmetry and adiabatic invariants are significant. Hence, many results about perturbation to symmetry and adiabatic invariants have been achieved in recent years[29-33]. Since El-Nabulsi's fractional model and adiabatic invariants have great theoretical and applied values, both still deserve further academic research.

Here we combine El-Nabulsi's fractional model with adiabatic invariants for the disturbed generalized Birkhoff system. Exact invariants are firstly presented on the basis of El-Nabulsi's fractional Noether theorem. And then adiabatic invariants for disturbed generalized El-Nabulsi's fractional Birkhoff system are given by investigating the perturbation to Noether symmetry. Finally, the Hojman-Urrutia problem[2] is discussed to illustrate the application of this method and its results.

1 Noether Symmetric Perturbation and Adiabatic Invariants for Generalized El-Nabulsi's Fractional Birkhoff System

In this section, one considers the adiabatic invariants of Noether symmetric perturbation and gives the main results for generalized El-Nabulsi's fractional Birkhoff system. Firstly, the equations for this system are given. Then, El-Nabulsi-Noether symmetric transformations and conservative quantities are introduced. After that, adiabatic invariants of Noether symmetric perturbation are presented.

1.1 Generalized El-Nabulsi-Birkhoff fractional equations

Generalized El-Nabulsi-Birkhoff fractional equations have the form[27]

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial \tau}= -\Lambda_\mu+\frac{1-\alpha}{t-\tau}R_\mu \tag{1}$$

where $\dot{a}^\nu=\dfrac{\mathrm{d}a^\nu}{\mathrm{d}\tau}$, $0<\alpha\leqslant 1$, τ is the intrinsic time, t the observer time, $\tau\neq t$, and $B=B(\tau, \boldsymbol{a})$ the Birkhoffian. $R_\mu=R_\mu(\tau, \boldsymbol{a})$ is the Birkhoff's function, and $\Lambda_\mu=\Lambda_\mu(\tau, \boldsymbol{a})$ the additional items. Both of them are C^2 functions with respect to all their arguments, $\mu,\nu=1, 2, \cdots, 2n$.

If $\alpha=1$, Eqs. (1) reduce to the standard generalized Birkhoff equations. If $\Lambda_\mu=0$, Eqs. (1) reduce to the El-Nabulsi-Birkhoff fractional equations.

1.2 El-Nabulsi-Noether symmetric transformations and conservative quantities

The El-Nabulsi-Pfaff action has the form[26]

$$S(\boldsymbol{a})=\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}[R_\mu(\tau, \boldsymbol{a})\dot{a}^\mu- B(\tau, \boldsymbol{a})](t-\tau)^{\alpha-1}\mathrm{d}\tau \tag{2}$$

Choose the infinitesimal transformations of r-parameter finite transformation group of τ and a^μ as

$$\bar{\tau}=\tau+\Delta\tau,\ \bar{a}^{\mu}(\bar{\tau})=a^{\mu}(\tau)+\Delta a^{\mu} \tag{3}$$

and their expanding forms are

$$\begin{gathered}\bar{\tau}=\tau+\varepsilon_{\sigma}\xi_{0}^{\sigma}(\tau,\ \boldsymbol{a})\\ \bar{a}^{\mu}(\bar{\tau})=a^{\mu}(\tau)+\varepsilon_{\sigma}\xi_{\mu}^{\sigma}(\tau,\ \boldsymbol{a})\end{gathered} \tag{4}$$

where $\varepsilon_{\sigma}(\sigma=1,\ 2,\ \cdots,\ r)$ are the infinitesimal parameters, and ξ_{0}^{σ}, ξ_{μ}^{σ} the infinitesimal generators of the infinitesimal transformations.

The basic formula for the variation of El-Nabulsi-Pfaff action[26] can be obtained by the transformations Eq. (4)

$$\begin{aligned}\Delta S=&\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\varepsilon_{\sigma}\Big\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(R_{\mu}\xi_{\mu}^{\sigma}-B\xi_{0}^{\sigma})\times\right.\\ &\left.(t-\tau)^{\alpha-1}\right]+\Big[\Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\Big)\dot{a}^{\nu}-\\ &\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial\tau}-\frac{1-\alpha}{t-\tau}R_{\mu}\Big](t-\tau)^{\alpha-1}\times\\ &(\xi_{\mu}^{\sigma}-\dot{a}^{\mu}\xi_{0}^{\sigma})\Big\}\,\mathrm{d}\tau\end{aligned} \tag{5}$$

If the following formula[27]

$$\Delta S=-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\Lambda_{\mu}(t-\tau)^{\alpha-1}\delta a^{\mu}\mathrm{d}\tau$$

holds for each of the infinitesimal transformations, the infinitesimal transformations are called the El-Nabulsi-Noether symmetric transformations. And one can verify the El-Nabulsi-Noether symmetry for the generalized El-Nabulsi′s fractional Birkhoff system.

If

$$\begin{aligned}\Delta S=&-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\Big[\frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta G)+\\ &\Lambda_{\mu}(t-\tau)^{\alpha-1}\delta a^{\mu}\Big]\mathrm{d}\tau\end{aligned}$$

where $G=G(\tau,\ \boldsymbol{a})$, then the infinitesimal transformations are called the El-Nabulsi-Noether quasi-symmetric transformations. Similarly, one can verify the El-Nabulsi-Noether quasi-symmetry for the generalized El-Nabulsi′s fractional Birkhoff system.

For the generalized El-Nabulsi′s fractional Birkhoff system, if the infinitesimal transformations of group (4) satisfy the following conditions[27]

$$\begin{aligned}&\Big(\frac{\partial R_{\mu}}{\partial\tau}\dot{a}^{\mu}-\frac{\partial B}{\partial\tau}\Big)\xi_{0}^{\sigma}+\Big(\frac{\partial R_{\mu}}{\partial a^{\nu}}\dot{a}^{\mu}-\frac{\partial B}{\partial a^{\nu}}\Big)\xi_{\nu}^{\sigma}+\\ &R_{\mu}(\dot{\xi}_{\mu}^{\sigma}-\dot{a}^{\mu}\dot{\xi}_{0}^{\sigma})+(R_{\mu}\dot{a}^{\mu}-B)\Big(\dot{\xi}_{0}^{\sigma}+\\ &\frac{1-\alpha}{t-\tau}\xi_{0}^{\sigma}\Big)+\Lambda_{\mu}(\xi_{\mu}^{\sigma}-\dot{a}^{\mu}\xi_{0}^{\sigma})=\\ &-\dot{G}^{\sigma}(t-\tau)^{1-\alpha}\quad\sigma=1,\ 2,\ \cdots,\ r\end{aligned} \tag{6}$$

Then there exist r linearly independent conservative quantities

$$\begin{gathered}I^{\sigma}=(R_{\mu}\xi_{\mu}^{\sigma}-B\xi_{0}^{\sigma})(t-\tau)^{\alpha-1}+G^{\sigma}=c^{\sigma}\\ \sigma=1,\ 2,\ \cdots,\ r\end{gathered} \tag{7}$$

When $\alpha=1$, one can attain Noether symmetry, Noether quasi-symmetry and the corresponding conservative quantities for the standard generalized Birkhoff system. When $\Lambda_{\mu}=0$, one can obtain the El-Nabulsi-Noether symmetry, the El-Nabulsi-Noether quasi-symmetry and the corresponding conservative quantities for the El-Nabulsi′s fractional Birkhoff system.

1.3 Noether symmetric perturbation and adiabatic invariants

Noether symmetric perturbation does not always lead to adiabatic invariants. In the sequel, one presents the conditions under which Noether symmetric perturbation can imply adiabatic invariants.

Definition[32] If $I_z=I_z(\tau,\ a^{\nu},\ \varepsilon)$ is a physical quantity for a mechanical system including ε in which the highest power is z, and its derivative with respect to τ is in direct proportion to ε^{z+1}, then I_z is called a z-th order adiabatic invariants of the mechanical system.

Specially, when $z=0$, one can get exact invariants. Hence

Theorem 1 If $\xi_{0}^{\sigma 0}$, $\xi_{\mu}^{\sigma 0}$ satisfy Eqs. (6) for the generalized El-Nabulsi′s fractional Birkhoff system, then there exist exact invariants

$$\begin{gathered}(R_{\mu}\xi_{\mu}^{\sigma 0}-B\xi_{0}^{\sigma 0})(t-\tau)^{\alpha-1}+G^{\sigma 0}=c^{\sigma 0}\\ \sigma=1,2,\cdots,r\end{gathered} \tag{8}$$

Suppose that the generalizd El-Nabulsi′s fractional Birkhoff system is perturbed by small quantities εQ_{μ}. Then the motion equations of the system become

$$\begin{aligned}&\Big(\frac{\partial R_{\nu}}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial a^{\nu}}\Big)\dot{a}^{\nu}-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial\tau}=\\ &-\Lambda_{\mu}+\frac{1-\alpha}{t-\tau}R_{\mu}+\varepsilon Q_{\mu}\end{aligned} \tag{9}$$

Under the action of small forces of perturbation εQ_{μ}, the previous symmetries and invariants of the system may vary. Assume that the perturbed generators ξ_{0}^{σ}, ξ_{μ}^{σ} $(\sigma=1,\ 2,\ \cdots,\ r)$ of infinitesimal transformations are small perturbation on the basis of the generators of symmetric transformations of an unperturbed system, then one has

$$\xi_0^\sigma = \xi_0^{\sigma 0} + \varepsilon \xi_0^{\sigma 1} + \varepsilon^2 \xi_0^{\sigma 2} + \cdots$$
$$\xi_\mu^\sigma = \xi_\mu^{\sigma 0} + \varepsilon \xi_\mu^{\sigma 1} + \varepsilon^2 \xi_\mu^{\sigma 2} + \cdots$$

In the meanwhile, due to the small perturbation, one also has

$$G^\sigma = G^{\sigma 0} + \varepsilon G^{\sigma 1} + \varepsilon^2 G^{\sigma 2} + \cdots$$

Theorem 2 For the generalized El-Nabulsi′s fractional Birkhoff system disturbed by small forces of perturbation εQ_μ, if there exists $G^{\sigma m}(\tau, \boldsymbol{a})$ $(m = 0,1,2,\cdots)$ such that the generators $\xi_0^{\sigma m}(\tau, \boldsymbol{a})$, $\xi_\mu^{\sigma m}(\tau, \boldsymbol{a})$ of the infinitesimal transformations satisfy

$$\left(\frac{\partial R_\mu}{\partial \tau}\dot{a}^\mu - \frac{\partial B}{\partial \tau}\right)\xi_0^{\sigma m} + \left(\frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\mu - \frac{\partial B}{\partial a^\nu}\right)\times \xi_\nu^{\sigma m} + R_\mu(\dot{\xi}_\mu^{\sigma m} - \dot{a}^\mu \dot{\xi}_0^{\sigma m}) + (R_\mu \dot{a}^\mu - B)\times \left(\dot{\xi}_0^{\sigma m} + \frac{1-\alpha}{t-\tau}\xi_0^{\sigma m}\right) + \Lambda_\mu(\xi_\mu^{\sigma m} - \dot{a}^\mu \xi_0^{\sigma m}) - Q_\mu(\xi_\mu^{\sigma(m-1)} - \dot{a}^\mu \xi_0^{\sigma(m-1)}) = -\dot{G}^{\sigma m}(t-\tau)^{1-\alpha} \tag{10}$$

Then the generalized El-Nabulsi′s fractional Birkhoff system has the z-th order adiabatic invariants

$$I_z^\sigma = \sum_{m=0}^{z} \varepsilon^m \left[(R_\mu \xi_\mu^{\sigma m} - B\xi_0^{\sigma m})\times (t-\tau)^{\alpha-1} + G^{\sigma m}\right] \tag{11}$$

where one sets $\xi_\mu^{\sigma(m-1)} = \xi_0^{\sigma(m-1)} = 0$, when $m = 0$.

Proof Differentiating I_z^σ with respect to τ, and considering Eqs. (9, 10), one has

$$\frac{\mathrm{d}}{\mathrm{d}\tau} I_z^\sigma = \sum_{m=0}^{z}\varepsilon^m\left[\left(\frac{\partial R_\mu}{\partial \tau} + \frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\nu\right)\xi_\mu^{\sigma m} + R_\mu \dot{\xi}_\mu^{\sigma m} - \left(\frac{\partial B}{\partial \tau} + \frac{\partial B}{\partial a^\nu}\dot{a}^\nu\right)\xi_0^{\sigma m} - B\dot{\xi}_0^{\sigma m} + (R_\mu \xi_\mu^{\sigma m} - B\xi_0^{\sigma m})\frac{1-\alpha}{t-\tau} + \frac{\dot{G}^{\sigma m}}{(t-\tau)^{\alpha-1}}\right](t-\tau)^{\alpha-1} =$$
$$(t-\tau)^{\alpha-1}\sum_{m=0}^{z}\varepsilon^m\left[\left(\frac{\partial R_\mu}{\partial \tau} + \frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\nu\right)\xi_\mu^{\sigma m} - \frac{\partial B}{\partial a^\nu}\dot{a}^\nu \xi_0^{\sigma m} + \frac{1-\alpha}{t-\tau}R_\mu \xi_\mu^{\sigma m} - \frac{\partial R_\mu}{\partial \tau}\dot{a}^\mu \xi_0^{\sigma m} - \left(\frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\mu - \frac{\partial B}{\partial a^\nu}\right)\xi_\nu^{\sigma m} - \frac{1-\alpha}{t-\tau}R_\mu \dot{a}^\mu \xi_0^{\sigma m} - \Lambda_\mu(\xi_\mu^{\sigma m} - \dot{a}^\mu \xi_0^{\sigma m}) + Q_\mu(\xi_\mu^{\sigma(m-1)} - \dot{a}^\mu \xi_0^{\sigma(m-1)})\right] = (t-\tau)^{\alpha-1}\sum_{m=0}^{z}\varepsilon^m\left\{\left[\frac{1-\alpha}{t-\tau}R_\mu + \frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial \tau} - \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \Lambda_\mu\right]\times (\xi_\mu^{\sigma m} - \dot{a}^\mu \xi_0^{\sigma m}) - \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu \dot{a}^\mu \xi_0^{\sigma m} + Q_\mu(\xi_\mu^{\sigma(m-1)} - \dot{a}^\mu \xi_0^{\sigma(m-1)})\right\} =$$
$$(t-\tau)^{\alpha-1}\sum_{m=0}^{z}\varepsilon^m\left[-\varepsilon Q_\mu(\xi_\mu^{\sigma m} - \dot{a}^\mu \xi_0^{\sigma m}) + Q_\mu(\xi_\mu^{\sigma(m-1)} - \dot{a}^\mu \xi_0^{\sigma(m-1)})\right] = -\varepsilon^{z+1}\frac{Q_\mu(\xi_\mu^{\sigma z} - \dot{a}^\mu \xi_0^{\sigma z})}{(t-\tau)^{1-\alpha}} \tag{12}$$

Hence, I_z^σ $(\sigma = 1, 2, \cdots, r)$ are the z-th order adiabatic invariants for the generalized El-Nabulsi′s fractional Birkhoff system.

As special cases, one can also obtain the following results.

Theorem 3[32] For the standard generalized Birkhoff system disturbed by small forces of perturbation εQ_μ, if there exists $G^{\sigma m}(\tau, \boldsymbol{a})$ such that the generators $\xi_0^{\sigma m}(\tau, \boldsymbol{a})$, $\xi_\mu^{\sigma m}(\tau, \boldsymbol{a})$ of the infinitesimal transformations satisfy

$$\left(\frac{\partial R_\mu}{\partial \tau}\dot{a}^\mu - \frac{\partial B}{\partial \tau}\right)\xi_0^{\sigma m} + \left(\frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\mu - \frac{\partial B}{\partial a^\nu}\right)\times \xi_\nu^{\sigma m} + R_\mu(\dot{\xi}_\mu^{\sigma m} - \dot{a}^\mu \dot{\xi}_0^{\sigma m}) + (R_\mu \dot{a}^\mu - B)\dot{\xi}_0^{\sigma m} - Q_\mu(\xi_\mu^{\sigma(m-1)} - \dot{a}^\mu \xi_0^{\sigma(m-1)}) + \Lambda_\mu(\xi_\mu^{\sigma m} - \dot{a}^\mu \xi_0^{\sigma m}) = -\dot{G}^{\sigma m} \tag{13}$$

Then the standard generalized Birkhoff system has the z-th order adiabatic invariants

$$I_z^\sigma = \sum_{m=0}^{z}\varepsilon^m(R_\mu \xi_\mu^{\sigma m} - B\xi_0^{\sigma m} + G^{\sigma m}) \tag{14}$$

where one sets $\xi_\mu^{\sigma(m-1)} = \xi_0^{\sigma(m-1)} = 0$, when $m = 0$.

Theorem 4[33] For the El-Nabulsi′s fractional Birkhoff system disturbed by small forces of perturbation εQ_μ, if there exists $G^{\sigma m}(\tau, \boldsymbol{a})$ $(m = 0, 1, 2, \cdots)$ such that the generators $\xi_0^{\sigma m}(\tau, \boldsymbol{a})$, $\xi_\mu^{\sigma m}(\tau, \boldsymbol{a})$ of the infinitesimal transformations satisfy

$$\left(\frac{\partial R_\mu}{\partial \tau}\dot{a}^\mu - \frac{\partial B}{\partial \tau}\right)\xi_0^{\sigma m} + \left(\frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\mu - \frac{\partial B}{\partial a^\nu}\right)\times \xi_\nu^{\sigma m} + R_\mu(\dot{\xi}_\mu^{\sigma m} - \dot{a}^\mu \dot{\xi}_0^{\sigma m}) + (R_\mu \dot{a}^\mu - B)\times \left(\dot{\xi}_0^{\sigma m} + \frac{1-\alpha}{t-\tau}\xi_0^{\sigma m}\right) - Q_\mu(\xi_\mu^{\sigma(m-1)} - \dot{a}^\mu \xi_0^{\sigma(m-1)}) = -\dot{G}^{\sigma m}(t-\tau)^{1-\alpha} \tag{15}$$

Then the El-Nabulsi′s fractional Birkhoff system has the z-th order adiabatic invariants

$$I_z^\sigma = \sum_{m=0}^{z}\varepsilon^m\left[(R_\mu \xi_\mu^{\sigma m} - B\xi_0^{\sigma m})\times (t-\tau)^{\alpha-1} + G^{\sigma m}\right] \tag{16}$$

where one sets $\xi_\mu^{\sigma(m-1)} = \xi_0^{\sigma(m-1)} = 0$, when $m = 0$.

2 An Illustrative Example

Consider a fourth order generalized El-Nabulsi's fractional Birkhoff system, whose Birkhoffian, Birkhoff's functions and the additional items are

$$B = \frac{1}{2}[(a^3)^2 + 2a^2a^3 - (a^4)^2]$$
$$R_1 = a^2 + a^3,\ R_2 = 0,\ R_3 = a^4$$
$$R_4 = 0,\ \Lambda_1 = \Lambda_2 = \Lambda_4 = 0$$
$$\Lambda_3 = a^1 + a^2 + a^4 \tag{17}$$

As an example[2], one tries to study its Noether symmetrical perturbation and adiabatic invariants.

Assume that ξ_0^0, ξ_μ^0, and G^0 satisfy Eqs. (6), that is,

$$(\dot{a}^1 - a^3)\xi_2^0 + (\dot{a}^1 - a^3 - a^2)\xi_3^0 + (\dot{a}^3 + a^4)\xi_4^0 + (a^2 + a^3)(\dot{\xi}_1^0 - \dot{a}^1\dot{\xi}_0^0) + a^4(\dot{\xi}_3^0 - \dot{a}^3\dot{\xi}_0^0) + (a^2\dot{a}^1 + a^3\dot{a}^1 + a^4\dot{a}^3 - B)\left(\dot{\xi}_0^0 + \frac{1-\alpha}{t-\tau}\xi_0^0\right) + (a^1 + a^2 + a^4)(\xi_3^0 - \dot{a}^3\xi_0^0) = -\dot{G}^0(t-\tau)^{1-\alpha} \tag{18}$$

Eq. (18) has one solution

$$\xi_0^0 = 0,\ \xi_1^0 = 1,\ \xi_2^0 = (t-\tau)^{1-\alpha}$$
$$\xi_3^0 = 0,\ \xi_4^0 = (t-\tau)^{1-\alpha},\ G^0 = 1 \tag{19}$$

Hence, one can obtain an exact invariant from Theorem 1

$$I_0 = (a^2 + a^3)(t-\tau)^{\alpha-1} + 1 = \text{const} \tag{20}$$

When the system corresponding to Eq. (17) is disturbed by

$$\varepsilon Q_1 = \varepsilon[(a^1)^2 + a^4(a^1 - a^2)]$$
$$\varepsilon Q_2 = 0,\ \varepsilon Q_3 = \varepsilon a^1,\ \varepsilon Q_4 = 0 \tag{21}$$

From Eqs. (10), one has

$$(\dot{a}^1 - a^3)\xi_2^1 + (\dot{a}^1 - a^3 - a^2)\xi_3^1 + (\dot{a}^3 + a^4)\xi_4^1 + (a^2 + a^3)(\dot{\xi}_1^1 - \dot{a}^1\dot{\xi}_0^1) + a^4(\dot{\xi}_3^1 - \dot{a}^3\dot{\xi}_0^1) + (a^2\dot{a}^1 + a^3\dot{a}^1 + a^4\dot{a}^3 - B)\left(\dot{\xi}_0^1 + \frac{1-\alpha}{t-\tau}\xi_0^1\right) + (a^1 + a^2 + a^4)(\xi_3^1 - \dot{a}^3\xi_0^1) - [(a^1)^2 + a^4(a^1 - a^2)](\xi_1^0 - \dot{a}^1\xi_0^0) - a^1(\xi_3^0 - \dot{a}^3\xi_0^0) = -\dot{G}^1(t-\tau)^{1-\alpha} \tag{22}$$

Eq. (22) has a solution

$$\xi_0^1 = 0,\ \xi_1^1 = a^3,\ \xi_2^1 = a^2$$
$$\xi_3^1 = a^1,\ \xi_4^1 = a^2,\ G^1 = 1 \tag{23}$$

Using Theorem 2, one can obtain the first order adiabatic invariant as follows

$$I_1 = (a^2 + a^3)(t-\tau)^{\alpha-1} + 1 + \varepsilon\{[(a^2 + a^3)a^3 + a^1a^4](t-\tau)^{\alpha-1} + 1\} \tag{24}$$

Furthermore, the higher order adiabatic invariants can also be obtained.

3 Conclusions

Noether symmetric perturbation and adiabatic invariants for the generalized El-Nabulsi's fractional Birkhoff system are investigated. Based on infinitesimal transformations, the exact invariants are given for the generalized El-Nabulsi's fractional Birkhoff system. Then the adiabatic invariants of Noether symmetric perturbation for the disturbed generalized El-Nabulsi's fractional Birkhoff system are obtained. The adiabatic invariants of Noether symmetric perturbation for generalized El-Nabulsi's fractional Birkhoff system are first studied. And the obtained results comprises Eq. (9) of disturbed generalized El-Nabulsi's fractional Birkhoff system, Eq. (10) of Noether symmetric perturbation and Theorem 2. These results present the perturbation and adiabatic invariants for generalized El-Nabulsi's fractional Birkhoff system. Since few researches are about perturbation of Birkhoff system due to its complexity, more such work can be done in depth. Moreover, based on some known results[3,34], it is considered that integration methods for generalized El-Nabulsi's fractional Birkhoff system also deserve further study.

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